

6

Foundations

Most of the sentences we will be interested in proving are not valid sentences of predicate logic, i.e., they are not true under every interpretation, but they are true under certain specific interpretations. For example, a sentence such as "For every integer x , $x + 0 = x$," which might be written in predicate logic as $(\forall x)p(f(x, a), x)$, is not valid, but it is certainly true under an interpretation over the integers that assigns

a to be 0

f to be the *addition* function

p to be the *equality* relation.

Our problem is to describe the interpretations under which we intend these sentences to be true. For this purpose, we introduce the general notion of a "theory."

6.1 DEFINITION OF A THEORY

The interpretations we are concerned with are determined by a set of closed sentences, the *axioms* of the theory.

Definition (axioms)

The *axioms* of a theory are a set of closed sentences

A_1, A_2, A_3, \dots

We will say that the theory is *defined* by its axioms. ┘

Note that we do not require that the given set of axioms be finite.

Example (family theory). Suppose we would like to define a theory of family relationships. In the naive “family” interpretation \mathcal{I} we have in mind, the domain is the set of people, and, intuitively speaking,

$f(x)$ means the father of x

$m(x)$ means the mother of x

$p(x, y)$ means y is a parent of x

$gf(x, y)$ means y is a grandfather of x

$gm(x, y)$ means y is a grandmother of x .

(More precisely, $p_{\mathcal{I}}(d, e)$ holds if e is a parent of d , and so forth.) We chose the names of the function symbols, f , m , and predicate symbols, p , gf , gm , to give them mnemonic associations. We may understand unconventional symbols, such as m , gf , and gm , to be informal notations for ordinary symbols of predicate logic such as g , q , and r .

The axioms of the theory are the following set of closed sentences:

$\mathcal{F}_1 : (\forall x)p(x, f(x))$	(<i>father</i>)
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That is, everyone’s father is his or her parent.

$\mathcal{F}_2 : (\forall x)p(x, m(x))$	(<i>mother</i>)
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That is, everyone’s mother is his or her parent.

$\mathcal{F}_3 : (\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } gf(x, f(y)) \end{array} \right]$	(<i>grandfather</i>)
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That is, the father of one’s parent is his or her grandfather.

$\mathcal{F}_4 : (\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } gm(x, m(y)) \end{array} \right]$	(<i>grandmother</i>)
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That is, the mother of one’s parent is his or her grandmother.

We surround these sentences with a box to indicate that they are axioms.

Let us consider the relationship between the axioms of a theory and a specific interpretation we have in mind.

Definition (model, validity, implication, equivalence, consistency)

Let \mathcal{A}_i be the axioms of a theory T .

An interpretation \mathcal{I} is a *model* for T if each axiom \mathcal{A}_i of the theory is true under \mathcal{I} .

A closed sentence \mathcal{F} is *valid* in T if \mathcal{F} is true under every model for T .

A sentence \mathcal{F} *implies* a sentence \mathcal{G} in T if, whenever \mathcal{F} is true under a model \mathcal{I} for T , \mathcal{G} is also true under \mathcal{I} .

Two sentences \mathcal{F} and \mathcal{G} are *equivalent* in T if \mathcal{F} and \mathcal{G} have the same truth-value under every model for T .

The theory T is *consistent* if there is at least one model for T . \blacksquare

When we speak of a *theory*, we mean its axioms, its models, and its valid sentences.

As an immediate consequence of the definition, we have that every axiom for a theory is valid in the theory. Also, if a theory is inconsistent, it has no models, and therefore every sentence is “vacuously” valid in the theory. [In the two-volume version of this book, we introduce a restricted vocabulary for each theory. Here each theory employs the entire vocabulary of predicate logic.]

In defining a theory, we make sure that the interpretation we have in mind is a model for the theory. In general, however, there are many models for a theory.

Example (models). The “family” interpretation we had in mind is a model for the family theory defined by the axioms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, and \mathcal{F}_4 in the example above because each of the axioms is true under the family interpretation. This is the *intended model* for the theory; however, there are many other models.

Consider the “number” interpretation \mathcal{I} over the domain of the nonnegative integers under which, intuitively speaking,

$$f(x) \quad \text{is} \quad 2x$$

$$m(x) \quad \text{is} \quad 3x$$

$$p(x, y) \quad \text{is} \quad y = 2x \quad \text{or} \quad y = 3x$$

$$gf(x, y) \quad \text{is} \quad y = 4x \quad \text{or} \quad y = 6x$$

$$gm(x, y) \quad \text{is} \quad y = 6x \quad \text{or} \quad y = 9x.$$

(More precisely, $f_{\mathcal{I}}(d)$ is $2d$, and so forth.) Each of the above axioms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, and \mathcal{F}_4 is true under the interpretation \mathcal{I} . For instance, the intuitive meaning of the *mother* axiom \mathcal{F}_2 ,

$$(\forall x)p(x, m(x)),$$

is

for every integer x ,

$$3x = 2x \quad \text{or} \quad 3x = 3x,$$

and the intuitive meaning of the *grandfather* axiom \mathcal{F}_3 ,

$$(\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } gf(x, f(y)) \end{array} \right],$$

is

$$\begin{array}{l} \text{for every integer } x \text{ and } y, \\ \text{if } y = 2x \text{ or } y = 3x, \\ \text{then } 2y = 4x \text{ or } 2y = 6x. \end{array}$$

Therefore the “number” interpretation \mathcal{I} is a model for the family theory. ┘

For a given closed sentence to be valid in a theory, it must be true under every model for the theory. To establish validity in a theory, we may apply the same techniques we used in predicate logic itself.

Example (validity). Suppose we would like to establish the validity of the sentence

$$\mathcal{F}: (\forall x)(\exists z)gm(x, z),$$

that is, everyone has a grandmother, in the above “family” theory. Let us give an informal argument based on the semantic rules and our common sense.

Let \mathcal{I} be an arbitrary model for the family theory. Then each of the axioms \mathcal{F}_1 , \mathcal{F}_2 , \mathcal{F}_3 , and \mathcal{F}_4 is true under \mathcal{I} .

Because the *father* axiom \mathcal{F}_1 , that is,

$$(\forall x)p(x, f(x)),$$

is true under \mathcal{I} , we know (by the \forall rule) that

for every domain element d ,

$$p(x, f(x))$$

is true under the modified interpretation $\langle x \leftarrow d \rangle \circ \mathcal{I}$.

Therefore (by the *application* rule), we know that

for every domain element d ,

$$(\dagger) \quad p(x, y)$$

is true under $\langle x \leftarrow d \rangle \circ \langle y \leftarrow f_{\mathcal{I}}(d) \rangle \circ \mathcal{I}$,

where $f_{\mathcal{I}}$ is the function assigned to f by \mathcal{I} .

Because the *grandmother* axiom \mathcal{F}_4 , that is,

$$(\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } gm(x, m(y)) \end{array} \right],$$

is true under \mathcal{I} , we know (by the \forall rule) that,

for all domain elements d and e , the subsentence

$$(\dagger) \quad \begin{array}{l} \text{if } p(x, y) \\ \text{then } gm(x, m(y)) \end{array}$$

is true under $\langle x \leftarrow d \rangle \circ \langle y \leftarrow e \rangle \circ \mathcal{I}$.

Consider an arbitrary domain element d . Taking e to be the domain element $f_{\mathcal{I}}(d)$, we have that

the implication (\dagger)

is true under the interpretation $\langle x \leftarrow d \rangle \circ \langle y \leftarrow f_{\mathcal{I}}(d) \rangle \circ \mathcal{I}$.

Because the implication (\dagger) and its antecedent (\dagger) , that is, $p(x, y)$, are both true under $\langle x \leftarrow d \rangle \circ \langle y \leftarrow f_{\mathcal{I}}(d) \rangle \circ \mathcal{I}$, its consequent

$$gm(x, m(y))$$

is also true under $\langle x \leftarrow d \rangle \circ \langle y \leftarrow f_{\mathcal{I}}(d) \rangle \circ \mathcal{I}$.

Therefore (by the *application* rule), the sentence

$$gm(x, z)$$

is true under $\langle x \leftarrow d \rangle \circ \langle z \leftarrow m_{\mathcal{I}}(f_{\mathcal{I}}(d)) \rangle \circ \mathcal{I}$.

Hence (by the \exists rule),

$$(\exists z)gm(x, z)$$

is true under $\langle x \leftarrow d \rangle \circ \mathcal{I}$.

Because d is an arbitrary domain element, we know (by the \forall rule) that the sentence \mathcal{F} , that is,

$$(\forall x)(\exists z)gm(x, z),$$

is true under \mathcal{I} .

Because \mathcal{I} was taken to be an arbitrary model for the family theory, this means that \mathcal{F} is valid in the family theory. \blacksquare

When we wish to distinguish ordinary predicate logic from an axiomatic theory, we shall refer to *pure* predicate logic. One may regard pure predicate logic as an axiomatic theory in which the set of axioms is empty.

Up to now we have been very careful to avoid confusing a symbol in a sentence and its value under an interpretation. For example, we never consider hybrid objects such as $f(a, d)$, in which f is a function symbol, a is a constant symbol, and d is a domain element. Such a construct is neither an expression in predicate logic nor an element in the domain of an interpretation. Our pedantry in this respect, unfortunately, has made our proofs of validity more cumbersome than necessary. Informal arguments may be made more concise and given more intuitive content if we agree to confuse symbols and their meanings under an intended interpretation. The argument in the above example, for instance, can be abbreviated if we say that x "is" a person and $f(x)$ "is" x 's father, even though we are confusing symbols and their meanings in this way. We shall call such a style of proof an "intuitive argument."

Example (intuitive argument). Suppose we would like to give an intuitive argument to establish again the validity of the sentence

$$\mathcal{F} : (\forall x)(\exists z)gm(x, z),$$

that is, "Everyone has a grandmother," in the family theory.

Consider an arbitrary person x . By the *father* axiom, \mathcal{F}_1 , we know that the father $f(x)$ of x is a parent of x ; that is,

$$(\dagger) \quad p(x, f(x)).$$

By the *grandmother* axiom, \mathcal{F}_4 , we have (taking y to be $f(x)$), that

$$(\ddagger) \quad \begin{array}{l} \text{if } p(x, f(x)) \\ \text{then } gm(x, m(f(x))) \end{array}.$$

Hence, by (\dagger) and (\ddagger) , we know that the mother $m(f(x))$ of the father $f(x)$ of x is a grandmother of x ; that is,

$$gm(x, m(f(x))).$$

Therefore, we know by predicate logic that

$$(\exists z)gm(x, z).$$

Because this has been shown to be true for an arbitrary person x , we can conclude

$$(\forall x)(\exists z)gm(x, z)$$

is a valid sentence of the family theory. Usually this step is omitted from intuitive arguments. ─

Because such intuitive arguments are shorter and easier to follow than arguments with explicit interpretations, we shall use them from now on, except in situations in which it is important to preserve the distinction between an expression and its meaning. Whenever such an intuitive argument is given, a precise proof could be substituted.

Remark (basic predicate-logic properties). Note that in the intuitive argument we have made use of basic properties of predicate logic without mentioning them. For example, from the *grandmother* axiom \mathcal{F}_4 ,

$$(\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } gm(x, m(y)) \end{array} \right]$$

we deduced (\ddagger) , that is,

$$\begin{array}{l} \text{if } p(x, f(x)) \\ \text{then } gm(x, m(f(x))). \end{array}$$

For this purpose, we implicitly appealed twice to the *universal* part of the *quantifier instantiation* proposition, first taking x to be x itself and then taking y to be $f(x)$.

Henceforth, we shall often appeal to such basic properties with no explicit indication. ─

In **Problem 6.1**, the reader is asked to give an intuitive argument in the family theory.

6.2 AUGMENTING THEORIES

The sentence \mathcal{F} ,

$$(\forall x)(\exists z)gm(x, z),$$

which we have established in the preceding example, will be true under any model for the family theory. In particular, it will be true under the “number” interpretation we gave earlier. The sentence \mathcal{F} then has the intuitive meaning

for every integer x ,
there exists an integer z such that
 $z = 6x$ or $z = 9x$.

In showing that the sentence is valid in the theory, we are showing that it is true under all the models of the theory at once.

Our family theory is “incomplete” in the sense that there are many properties of family relationships that are not valid in the theory. For example, we cannot show the validity of

$$\mathcal{G}: (\forall x)[\text{not } p(x, x)],$$

that is, no one is his or her own parent. Even though this sentence is true under the “family” interpretation we have in mind, it is not true under all models of the theory. In particular, it is not true under the “number” interpretation, for which it has the intuitive meaning

for every integer x ,
it is not so that
 $x = 2x$ or $x = 3x$.

In fact, this sentence is false when x is taken to be 0.

If we want to develop a theory in which \mathcal{G} is valid, we can add \mathcal{G} to the axioms, obtaining an augmented theory defined by the axioms

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \text{ and } \mathcal{G}.$$

The “family” interpretation would still be a model for this new theory, but the “number” interpretation would not.

On the other hand, if we have the “number” interpretation in mind, we may consider adding ($\text{not } \mathcal{G}$), that is,

$$\text{not } (\forall x)[\text{not } p(x, x)],$$

or, equivalently,

$$\mathcal{G}' : (\exists x)p(x, x),$$

as an axiom, obtaining an alternative theory defined by

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \text{ and } \mathcal{G}'.$$

The “family” interpretation would not be a model for this theory, since no one is his or her own parent, but the “number” interpretation would be a model.

Note that, in adding new axioms to a theory, we may reduce its collection of models. In particular, if a new axiom is not true under one of the original models, that interpretation will not be a model for the augmented theory. Therefore, if a sentence is valid in the original theory, it is also valid in the augmented theory, but there may be some sentences that are not valid in the original theory but that are valid in the augmented theory.

In forming or augmenting a theory, we should be careful that the axioms are consistent, i.e., that there is at least one model for the theory.

Example (inconsistency). In our original formulation of the “family” theory defined by the axioms

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \text{ and } \mathcal{F}_4,$$

we did not account for the possibility of “first” people such as Adam. We might be tempted to add to our theory an axiom

$$\mathcal{A}: (\forall y)[\text{not } p(a, y)],$$

that is, person a has no parents. However, the augmented theory defined by the axioms

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \text{ and } \mathcal{A}$$

is inconsistent; i.e., there is no model for this theory.

To see this, suppose \mathcal{I} is a model for the augmented theory. Then the *father* axiom \mathcal{F}_1 , that is,

$$(\forall x)p(x, f(x)),$$

is true under \mathcal{I} . Therefore (taking x to be a)

$$p(a, f(a))$$

is true under \mathcal{I} . Thus (taking y to be $f(a)$)

$$(\exists y)p(a, y)$$

is also true under \mathcal{I} .

Note that this sentence is equivalent (by the duality between the quantifiers) to the sentence

$$\text{not } (\forall y)[\text{not } p(a, y)],$$

which is exactly the negation of the new axiom \mathcal{A} . Hence this axiom cannot be true under \mathcal{I} , contradicting our original supposition that \mathcal{I} is a model for the augmented theory. \blacksquare

If a theory is inconsistent, it has no models, and therefore every closed sentence is (vacuously) valid in the theory. For this reason, inconsistent theories are not very interesting. By demonstrating the existence of a model for a given set of axioms, we can ensure that the theory it defines is consistent.

We introduce now two axiomatic theories that are of importance in their own right.

6.3 THEORY OF STRICT PARTIAL ORDERINGS

For a given binary predicate symbol p , the *theory of the strict partial ordering* p is the theory defined by the axioms

$$S_1 : (\forall x, y, z) \left[\begin{array}{l} \text{if } p(x, y) \text{ and } p(y, z) \\ \text{then } p(x, z) \end{array} \right] \quad (\text{transitivity})$$

$$S_2 : (\forall x) [\text{not } p(x, x)] \quad (\text{irreflexivity})$$

Under any model for the theory defined by S_1 and S_2 , we shall say that p denotes a *strict partial ordering*.

In this theory, we shall use the conventional infix notation $x \prec y$ rather than $p(x, y)$. We can thus rewrite S_1 and S_2 as

$$S_1 : (\forall x, y, z) \left[\begin{array}{l} \text{if } x \prec y \text{ and } y \prec z \\ \text{then } x \prec z \end{array} \right] \quad (\text{transitivity})$$

$$S_2 : (\forall x) [\text{not } (x \prec x)] \quad (\text{irreflexivity})$$

The reader should understand that here $x \prec y$ is merely an informal notation for $p(x, y)$.

Let us consider two models for the theory of the strict partial ordering \prec .

Examples (strict partial orderings)

- *The less-than relation*

Consider an interpretation \mathcal{I} over the integers that assigns the less-than relation $<$ to the binary predicate symbol \prec . Then \mathcal{I} is a model for the theory of the strict partial ordering \prec , because the *transitivity* and *irreflexivity* axioms for \prec both hold under \mathcal{I} . The intuitive meanings of these axioms under this interpretation are

for every integer d_1, d_2 , and d_3 ,
 if $d_1 < d_2$ and $d_2 < d_3$
 then $d_1 < d_3$

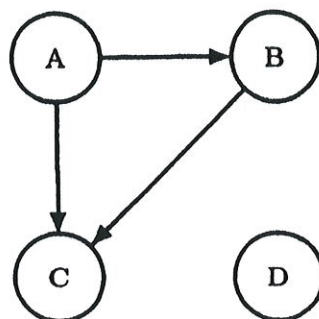
and

for every integer d ,
 not $d < d$,

which are both true.

• *A finite relation*

Consider an interpretation \mathcal{I} over the finite domain $\{A, B, C, D\}$ that assigns to \prec the binary relation illustrated by the following diagram:



In this representation, an arc leading directly from one element d to another e indicates that the relation holds between these elements; that is, $d \prec_{\mathcal{I}} e$ is true. Thus we have

$$A \prec_{\mathcal{I}} B, \quad B \prec_{\mathcal{I}} C, \quad \text{and} \quad A \prec_{\mathcal{I}} C.$$

The absence of an arc indicates that the relation does not hold between the corresponding elements. Thus

$$\text{not } B \prec_{\mathcal{I}} D, \quad \text{not } B \prec_{\mathcal{I}} B, \quad \text{not } B \prec_{\mathcal{I}} A, \quad \text{and so forth.}$$

The reader may confirm that the *transitivity* and *irreflexivity* axioms for \prec do hold under \mathcal{I} ; therefore \mathcal{I} is a model for the theory of the strict partial ordering \prec . \blacksquare

Now let us consider two interpretations that are not models for the theory of the strict partial ordering \prec .

Examples (not strict partial orderings).

• *The inequality relation*

Consider an interpretation \mathcal{I} over the integers that assigns the inequality relation \neq to the binary predicate symbol \prec . Then \mathcal{I} is not a model for the theory of the strict partial ordering \prec . The *irreflexivity* axiom for \prec does hold under \mathcal{I} ; its intuitive meaning is

$$\begin{aligned} &\text{for every integer } d, \\ &\quad \text{not } d \neq d, \end{aligned}$$

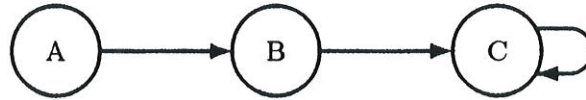
which is true. On the other hand, the *transitivity* axiom for \prec does not hold under \mathcal{I} ; its intuitive meaning is

$$\begin{aligned} &\text{for every integer } d_1, d_2, \text{ and } d_3, \\ &\quad \text{if } d_1 \neq d_2 \text{ and } d_2 \neq d_3 \\ &\quad \text{then } d_1 \neq d_3, \end{aligned}$$

which is false if d_1 and d_3 are the same integer and d_2 is a different integer.

• *A finite relation*

Consider an interpretation \mathcal{I} over the domain $\{A, B, C\}$ that assigns to \prec the binary relation illustrated by the following diagram:



Then \mathcal{I} is not a model for the theory of the strict partial ordering \prec . The *transitivity* axiom for \prec does not hold under \mathcal{I} , for we have

$$A \prec_{\mathcal{I}} B \quad \text{and} \quad B \prec_{\mathcal{I}} C \quad \text{but not} \quad A \prec_{\mathcal{I}} C.$$

Also, the *irreflexivity* axiom for \prec does not hold, for we have

$$C \prec_{\mathcal{I}} C. \quad \blacksquare$$

In **Problem 6.2**, the reader is asked to construct interpretations for the theory of the strict partial ordering \prec over a finite domain under which one of the axioms is true and the other is not.

ASYMMETRY

Now let us establish the validity of a sentence in the theory.

Proposition (asymmetry of strict partial orderings)

In the theory of the strict partial ordering \prec , the sentence

$$S: \quad (\forall x, y) \left[\begin{array}{l} \text{if } x \prec y \\ \text{then not } (y \prec x) \end{array} \right] \quad (\text{asymmetry})$$

is valid. \blacksquare

This means that S is true under all models for the theory defined by S_1 and S_2 . In other words, for any interpretation under which S_1 and S_2 are true, S is also true. We give an intuitive argument.

Proof. Suppose that, contrary to the *asymmetry* sentence S , there exist elements x and y such that both $x \prec y$ and $y \prec x$. Then, by the *transitivity* axiom S_1 , we have $x \prec x$. But this contradicts the *irreflexivity* axiom S_2 . \blacksquare

Remark (asymmetry implies irreflexivity). We have established that, in the theory defined by the *transitivity* axiom S_1 and the *irreflexivity* axiom S_2 , the *asymmetry* sentence S is valid.

On the other hand, note that the *asymmetry* sentence S ,

$$(\forall x, y) \left[\begin{array}{l} \text{if } x \prec y \\ \text{then not } (y \prec x) \end{array} \right],$$

by itself implies the *irreflexivity* sentence S_2 . For, taking y to be x , we obtain

$$(\forall x) \left[\begin{array}{l} \text{if } x \prec x \\ \text{then not } (x \prec x) \end{array} \right].$$

But (by propositional logic)

$$\begin{array}{l} \text{if } x \prec x \\ \text{then not } (x \prec x) \end{array} \quad \text{is equivalent to} \quad \text{not } (x \prec x).$$

Therefore (by the substitutivity of equivalence)

$$(\forall x) \left[\begin{array}{l} \text{if } x \prec x \\ \text{then not } (x \prec x) \end{array} \right]$$

is equivalent to

$$(\forall x) [\text{not } (x \prec x)],$$

which is the *irreflexivity* sentence S_2 . \blacksquare

Up to now we have been discussing a theory whose only axioms are the *transitivity* axiom S_1 and the *irreflexivity* axiom S_2 . Consider a theory in which these properties are true for some binary predicate symbol q . In other words, whatever the axioms of the theory are, the sentences

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } q(x, y) \text{ and } q(y, z) \\ \text{then } q(x, z) \end{array} \right]$$

and

$$(\forall x) [\text{not } q(x, x)]$$

are valid. We shall say that, in such a theory, q denotes a *strict partial ordering*. Of course, it is possible to have a theory with many binary predicate symbols, each denoting a strict partial ordering.

INVERSE RELATION

In practice, people often use the sentence $x \succ y$ synonymously with $y \prec x$. This can be reflected in our theory by introducing a new axiom. More precisely, we augment the theory of the strict partial ordering \prec by adding an axiom that defines the relation \succ , the *inverse* of \prec , as follows:

$S_3 : (\forall x, y) [x \succ y \equiv y \prec x]$	<i>(inverse)</i>
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As before, $x \succ y$ is merely an informal notation for an ordinary predicate symbol such as $q_{17}(x, y)$. We shall refer to S_3 as the *definition* of the inverse relation.

Whenever we augment a theory by introducing a new axiom, we run the risk of making our theory inconsistent. If so, the augmented theory will have no model, and therefore any sentence will be valid. We can show that the augmented theory is consistent by exhibiting a model for the enlarged axiom set. In this case, it is clear that there do exist models for the augmented theory obtained by introducing the new axiom S_3 into the strict partial ordering theory.

6.4 THEORY OF EQUIVALENCE RELATIONS

For any binary predicate symbol p , the *theory of the equivalence relation* p is the theory defined by the axioms:

$$Q_1 : (\forall x, y, z) \left[\begin{array}{l} \text{if } p(x, y) \text{ and } p(y, z) \\ \text{then } p(x, z) \end{array} \right] \quad (\text{transitivity})$$

$$Q_2 : (\forall x, y) \left[\begin{array}{l} \text{if } p(x, y) \\ \text{then } p(y, x) \end{array} \right] \quad (\text{symmetry})$$

$$Q_3 : (\forall x)p(x, x) \quad (\text{reflexivity})$$

Under any model for the theory defined by Q_1 , Q_2 , and Q_3 , we shall say that p denotes an *equivalence relation*.

The convention for an equivalence relation is to write $x \approx y$ rather than $p(x, y)$. In other words, we shall use the symbol \approx informally, rather than the predicate symbol p , to denote a relation for which Q_1 , Q_2 , and Q_3 hold. We shall thus write Q_1 , Q_2 , and Q_3 as

$Q_1 : (\forall x, y, z) \left[\begin{array}{l} \text{if } x \approx y \text{ and } y \approx z \\ \text{then } x \approx z \end{array} \right]$	(transitivity)
$Q_2 : (\forall x, y) \left[\begin{array}{l} \text{if } x \approx y \\ \text{then } y \approx x \end{array} \right]$	(symmetry)
$Q_3 : (\forall x)[x \approx x]$	(reflexivity)

Let us consider some models for the theory of the equivalence relation \approx .

Examples (equivalence relations).

• The congruence-modulo-2 relation

Consider an interpretation \mathcal{I} over the integers that assigns to \approx the “congruence modulo 2” relation \approx_2 , that is, for every integer d_1 and d_2 ,

$$d_1 \approx_2 d_2$$

if and only if

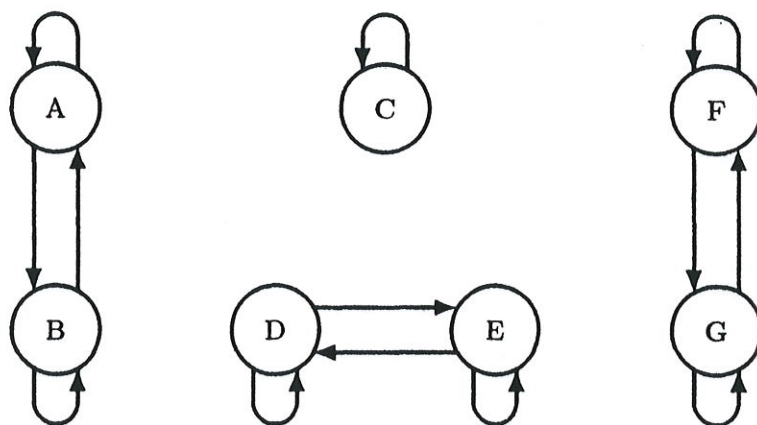
$$\left[\begin{array}{c} d_1 \text{ and } d_2 \text{ are both even} \\ \text{or} \\ d_1 \text{ and } d_2 \text{ are both odd} \end{array} \right].$$

Thus $2 \approx_2 6$ but not $1 \approx_2 2$.

The reader may confirm that the *transitivity*, *symmetry*, and *reflexivity* axioms Q_1 , Q_2 , and Q_3 are true under this interpretation.

• *A finite relation*

Consider an interpretation I over the domain $\{A, B, C, D, E, F, G\}$ that assigns \approx to be the binary relation illustrated by the following diagram:



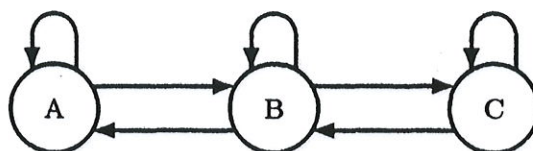
The reader may confirm that the *transitivity*, *symmetry*, and *reflexivity* axioms Q_1 , Q_2 , and Q_3 each hold under the interpretation I ; therefore I is a model for the theory of the equivalence relation \approx . \blacksquare

Now let us consider some finite interpretations that are not models for the theory of the equivalence relation \approx .

Examples (nonequivalence relations).

• *A nontransitive relation*

Over the domain $\{A, B, C\}$, consider the interpretation I that assigns to \approx the binary relation \approx_I illustrated by the following diagram:



Then I is not a model for the theory of the equivalence relation \approx , because the *transitivity* axiom Q_1 does not hold under I . In particular, we have

$$A \approx_I B \quad \text{and} \quad B \approx_I C \quad \text{but not} \quad A \approx_I C.$$

The reader may confirm that the *symmetry* and *reflexivity* axioms do hold under \mathcal{I} .

• *A nonsymmetric relation*

Over the domain $\{A, B\}$, consider the interpretation \mathcal{I} that assigns to \approx the binary relation $\approx_{\mathcal{I}}$ illustrated by the following diagram:



Then \mathcal{I} is not a model for the theory of the equivalence relation \approx , because the *symmetry* axiom Q_2 does not hold under \mathcal{I} ; we have

$$A \approx_{\mathcal{I}} B \quad \text{but not} \quad B \approx_{\mathcal{I}} A.$$

The reader may confirm that the *transitivity* and *reflexivity* axioms do hold under \mathcal{I} .

• *A nonreflexive relation*

Over the domain $\{A\}$, consider the interpretation \mathcal{I} that assigns to \approx the binary relation $\approx_{\mathcal{I}}$ illustrated by the following diagram:



In other words, $\approx_{\mathcal{I}}$ is the empty relation, which holds between no domain elements at all. Then \mathcal{I} is not a model for the theory of the equivalence relation \approx , because the *reflexivity* axiom Q_3 does not hold under $\approx_{\mathcal{I}}$; we have

$$\text{not } A \approx_{\mathcal{I}} A.$$

The reader may note that the *transitivity* and *symmetry* axioms Q_1 and Q_2 do hold under \mathcal{I} , because their antecedents are always false under \mathcal{I} . \blacksquare

The above three examples illustrate that the axioms for the equivalence relation \approx are *independent*; in other words, none of them is implied by the other two. For in each example we presented an interpretation under which two of the axioms are true and the third is false. If the two axioms implied the third, all three axioms would be true.

In **Problem 6.3**, the reader is asked to find the bug in a fallacious proof that one of the axioms for the theory of equivalence is implied by the other two.

DOUBLE TRANSITIVITY

From the axioms for the equivalence relation \approx , we can show the following result.

Proposition (double transitivity)

The sentence

$$(\forall u, v, x, y) \left[\begin{array}{l} \text{if } u \approx v \text{ and} \\ u \approx x \text{ and} \\ v \approx y \\ \text{then } x \approx y \end{array} \right] \quad (\text{double transitivity})$$

is valid in the theory of the equivalence relation \approx . \blacksquare

We give an intuitive justification.

Proof. Suppose that for arbitrary elements u, v, x , and y ,

$$u \approx v, \quad u \approx x, \quad \text{and} \quad v \approx y$$

are all true; we attempt to show that then

$$x \approx y$$

is true.

Because $u \approx v$ and $v \approx y$, we have, by the *transitivity* axiom Q_1 , that

$$u \approx y.$$

Because $u \approx x$, we have, by the *symmetry* axiom Q_2 , that

$$x \approx u.$$

Finally, because $x \approx u$ and $u \approx y$, we have, by the *transitivity* axiom Q_1 again, that

$$x \approx y,$$

which is the desired conclusion. \blacksquare

6.5 THEORY OF EQUALITY

The equality relation is an important tool that requires special treatment. We want to define a theory of equality under whose intended models a binary predicate symbol p is assigned the equality relation over the domain; i.e., the sentence $p(t_1, t_2)$ is true under a model \mathcal{I} if and only if the terms t_1 and t_2 have the same value under \mathcal{I} .

The usual convention is to write $x = y$ rather than $p(x, y)$ to denote the equality relation in the theory of equality. The reader should understand that here $x = y$ is merely an informal notation for $p(x, y)$, where p is a binary predicate symbol. It is not to be confused with our use of the notation $d_1 = d_2$, in giving the intuitive meanings for sentences, to indicate that d_1 and d_2 are the same domain elements.

THE THEORY

Although our previous theories have been defined by finite sets of axioms, the theory of equality requires a possibly infinite axiom set.

• Basic axioms

$$\mathcal{E}_1 : (\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \text{ and } y = z \\ \text{then } x = z \end{array} \right] \quad (\text{transitivity})$$

$$\mathcal{E}_2 : (\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } y = x \end{array} \right] \quad (\text{symmetry})$$

$$\mathcal{E}_3 : (\forall x)[x = x] \quad (\text{reflexivity})$$

• Substitutivity axiom schemata

For every k -ary function symbol f and for each i from 1 through k ,

$$\mathcal{E}_4(f) : (\forall z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_k) (\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } f(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_k) = \\ \quad f(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_k) \end{array} \right] \\ (\text{functional substitutivity for } f)$$

For every ℓ -ary predicate symbol q (other than $=$) and for each j from 1 through ℓ ,

$$\mathcal{E}_5(q) : (\forall z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_\ell) (\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } q(z_1, \dots, z_{j-1}, x, z_{j+1}, \dots, z_\ell) \\ \quad \equiv \\ \quad q(z_1, \dots, z_{j-1}, y, z_{j+1}, \dots, z_\ell) \end{array} \right] \\ (\text{predicate substitutivity for } q)$$

Thus both the *functional-substitutivity* axiom schema $\mathcal{E}_4(f)$ and the *predicate-substitutivity* axiom schema $\mathcal{E}_5(q)$ actually represent infinite sets of axioms, one or more for each function symbol f and predicate symbol q . We exclude the instances of the *predicate-substitutivity* axiom for $=$, in which q is the equality symbol $=$, because these instances follow from the other axioms.

Example. For a binary function symbol g , the corresponding instances of the *functional-substitutivity* axiom schema $\mathcal{E}_4(g)$ are

$$(\forall x, y, z_2) \left[\begin{array}{l} \text{if } x = y \\ \text{then } g(x, z_2) = g(y, z_2) \end{array} \right]$$

and

$$(\forall x, y, z_1) \left[\begin{array}{l} \text{if } x = y \\ \text{then } g(z_1, x) = g(z_1, y) \end{array} \right].$$

For a unary predicate symbol p , the corresponding instance of the *predicate-substitutivity* axiom schema $\mathcal{E}_5(p)$ is

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } p(x) \equiv p(y) \end{array} \right]. \quad \text{┘}$$

Note that the axioms for the theory of equality include the *transitivity*, *symmetry*, and *reflexivity* axioms from the theory of equivalence relations; in other words, $=$ denotes an equivalence relation. This means that since the *double-transitivity* property,

$$(\forall u, v, x, y) \left[\begin{array}{l} \text{if } u = v \text{ and} \\ \quad u = x \text{ and} \\ \quad v = y \\ \text{then } x = y \end{array} \right],$$

was proved to be valid in the theory of the equivalence relation $=$, it is also valid in the theory of equality.

The models for the theory of equality exhibit the following property.

Proposition (semantic rule for equality)

Suppose that \mathcal{I} is a model for the theory of equality and t_1 and t_2 are terms.

If

the value of t_1 under \mathcal{I}
is the same as
the value of t_2 under \mathcal{I} ,

then

$t_1 = t_2$ is true under \mathcal{I} . ┘

We shall refer to this result as the “ $=$ rule.”

Proof. Suppose that the terms t_1 and t_2 each have the same value, the domain element d , under \mathcal{I} . We would like to show that the sentence $t_1 = t_2$ is true under \mathcal{I} .

Let $=_{\mathcal{I}}$ be the binary relation assigned to the equality predicate symbol $=$ under \mathcal{I} . Then (by the *proposition* semantic rule, because the values of t_1 and t_2 under \mathcal{I} are each d)

the value of $t_1 = t_2$ under \mathcal{I} is $d =_{\mathcal{I}} d$.

We would like to show that

$$d =_{\mathcal{I}} d \text{ is true.}$$

We have assumed that \mathcal{I} is a model for the theory of equality. In particular, the *reflexivity* axiom \mathcal{E}_3 ,

$$(\forall x)[x = x],$$

is true under \mathcal{I} . Hence (taking x to be t_1), the sentence

$$t_1 = t_1 \text{ is true under } \mathcal{I}.$$

We know (according to the *proposition* semantic rule, because the value of t_1 under \mathcal{I} is d) that

$$\text{the value of } t_1 = t_1 \text{ under } \mathcal{I} \text{ is } d =_{\mathcal{I}} d.$$

Therefore

$$d =_{\mathcal{I}} d \text{ is true,}$$

as we wanted to show. \blacksquare

We wanted to formulate a theory of equality under whose models the binary predicate symbol $=$ would be assigned the equality relation over the domain; i.e., under any model \mathcal{I} for the theory, the sentence $t_1 = t_2$ is true under \mathcal{I} if and only if the terms t_1 and t_2 have the same value under \mathcal{I} . In fact, this is not the case for the theory of equality we have formulated.

The above semantic rule establishes that the implication holds in one direction; i.e., if the terms t_1 and t_2 have the same value under \mathcal{I} , then the sentence $t_1 = t_2$ is true under \mathcal{I} .

The converse of the implication, however, is not true: There are some “abnormal” models \mathcal{I} for the theory of equality such that some terms t_1 and t_2 have distinct values under \mathcal{I} , but the sentence $t_1 = t_2$ is true under \mathcal{I} nevertheless. This is illustrated by the following example.

Example (abnormal model for equality). Consider an interpretation \mathcal{I} over the domain $\{A, B\}$ of two elements that assigns

a to be A

b to be B

and each predicate symbol (including the equality symbol $=$) to be the relation that is true for all domain elements. (We do not care what functions are assigned to function symbols under \mathcal{I} .)

This is a model for the theory of equality; each of the axioms \mathcal{E}_1 through \mathcal{E}_5 is true under \mathcal{I} . For instance, to show that the *symmetry* axiom \mathcal{E}_2 is true under \mathcal{I} , it suffices to show that, for every domain element d and e , the subsentence

$$\begin{array}{l} \text{if } x = y \\ \text{then } y = x \end{array}$$

is true under the modified interpretation $\langle x \leftarrow d \rangle \circ \langle y \leftarrow e \rangle \circ \mathcal{I}$. But since the equality symbol $=$ is assigned the binary relation that is true for all domain elements, including e and d , the consequent $y = x$ of the implication, and hence the implication itself, is true under the modified interpretation.

The truth under \mathcal{I} of the other axioms for equality may be established similarly. Under \mathcal{I} , the terms a and b have distinct values A and B , respectively, but the sentence $a = b$ is true. \blacksquare

The above example illustrates that the converse of the semantic rule for equality (the $=$ rule) is not true. In other words, there are some models for the theory of equality under which the equality predicate symbol $=$ is not assigned the normal equality relation over the domain. Such "abnormal" models cannot be avoided in predicate logic, but they do not disturb us because the sentences we shall want to prove concerning the equality relation will be true under the abnormal models as well as the normal models.

In **Problem 6.4**, the reader is asked to show that a certain sentence is not valid in the theory of equality.

SUBSTITUTIVITY OF EQUALITY

The most important property of the theory of equality is given in the following proposition.

Proposition (substitutivity of equality)

Suppose s , t , and $r\langle s \rangle$ are terms; then the universal closure of

$$\begin{array}{ll} \text{if } s = t & \\ \text{then } r\langle s \rangle = r\langle t \rangle & (\text{term}) \end{array}$$

is valid in the theory of equality.

Suppose s and t are terms and $\mathcal{F}\langle s \rangle$ is a sentence; then the universal closure of

$$\begin{array}{ll} \text{if } s = t & \\ \text{then } \mathcal{F}\langle s \rangle \equiv \mathcal{F}\langle t \rangle & (\text{sentence}) \end{array}$$

is valid in the theory of equality. \blacksquare

Recall that, for any term $r\langle s \rangle$, the term $r\langle t \rangle$ denotes the result of safely replacing zero, one, or more free occurrences of s in $r\langle s \rangle$ with t . Similarly for sentences $\mathcal{F}\langle s \rangle$ and $\mathcal{F}\langle t \rangle$.

Let us consider an example.

Example. According to the *term* part of the proposition, the sentence

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = f(y) \\ \text{then } h(g(x, x)) = h(g(f(y), x)) \end{array} \right]$$

is valid in the theory of equality, because $h(g(f(y), x))$ is the result of safely replacing one of the free occurrences of x in $h(g(x, x))$ with $f(y)$.

Also, according to the *sentence* part of the proposition, the sentence

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = f(y) \\ \text{then } \left[\begin{array}{l} (\exists y)p(x, y) \\ \equiv \\ (\exists y')p(f(y), y') \end{array} \right] \end{array} \right]$$

is valid in the theory of equality, because $(\exists y')p(f(y), y')$ is the result of safely replacing the free occurrence of x in $(\exists y)p(x, y)$ with $f(y)$. Note that we have renamed the bound variable y to y' . \blacksquare

The truth of the proposition is intuitively clear, but the general proof, which we omit, is rather technical.

REPLACEMENT

Using the *substitutivity-of-equality* proposition one can establish another important property of the theory of equality.

Proposition (replacement)

Suppose x is a variable, t is a term, and $\mathcal{F}[x]$ is a sentence, where x does not occur free in t .

Then

$$(\forall x)[\text{if } x = t \text{ then } \mathcal{F}[x]] \text{ is equivalent to } \mathcal{F}[t] \quad (\text{universal})$$

and

$$(\exists x)[x = t \text{ and } \mathcal{F}[x]] \text{ is equivalent to } \mathcal{F}[t] \quad (\text{existential})$$

in the theory of equality. \blacksquare

One can also establish a more general version of the proposition, in which n variables x_1, x_2, \dots, x_n in a sentence $\mathcal{F}[x_1, x_2, \dots, x_n]$ are replaced by n terms t_1, t_2, \dots, t_n .

The *replacement* proposition would not hold if we used the partial substitution operation instead of the total substitution operation or if we abolished the

restriction that x does not occur free in t ; the reader is requested to show this in **Problem 6.5**.

The reader is also requested (in **Problem 6.6**) to show the validity of the following properties of conditional terms in the theory of equality:

$$(\text{if true then } a \text{ else } b) = a \quad (\text{true})$$

$$(\text{if false then } a \text{ else } b) = b \quad (\text{false})$$

$$(\forall x) \left[\begin{array}{c} f(\text{if } p(x) \text{ then } a \text{ else } b) \\ = \\ \text{if } p(x) \text{ then } f(a) \text{ else } f(b) \end{array} \right] \quad (\text{distributivity})$$

THEORY WITH EQUALITY

Often we wish to define theories whose models assign special meanings to certain constant, function, and predicate symbols in addition to the equality predicate symbol $=$. For this purpose we may provide special axioms for the theory, as well as the equality axioms \mathcal{E}_1 through \mathcal{E}_5 . A theory that is defined by a set of axioms that includes the equality axioms is called a *theory with equality*.

In general, when we describe a model \mathcal{I} for a theory with equality, we will assume (unless stated otherwise) that the equality symbol $=$ is assigned the normal equality relation, that is,

$$x = y \text{ is true under } \mathcal{I} \text{ if and only if } x_{\mathcal{I}} \text{ is identical to } y_{\mathcal{I}}.$$

Under such a model all the equality axioms are satisfied.

In any theory with equality there is exactly one equality relation. This is expressed precisely in the following result. (The proof is requested in **Problem 6.7**.)

Proposition (uniqueness of equality)

In any theory with equality, let $r(x, y)$ be an equality symbol, i.e., a binary predicate symbol such that the equality axioms \mathcal{E}_1 through \mathcal{E}_5 are valid for r . (In other words, r satisfies the *transitivity*, *symmetry*, and *reflexivity* axioms and the *functional* and *predicate-substitutivity* axiom schemata.)

Then r and the equality symbol $=$ denote the same relation; i.e., the sentence

$$(\forall x, y) [r(x, y) \equiv (x = y)] \quad (\text{uniqueness})$$

is valid in the theory. \blacksquare

We have seen that there are abnormal models for the theory of equality, under which an equality symbol is not assigned the normal equality relation over the domain. The above proposition establishes that under any one model for the theory of equality, all the equality symbols must be assigned the same relation, even if that relation is not the normal one.

In the following sections, we give three examples of theories with equality: the theories of weak partial orderings, groups, and pairs.

6.6 THEORY OF WEAK PARTIAL ORDERINGS

Our first example of a theory with equality is the *theory of the weak partial ordering* \preceq , where \preceq is any binary predicate symbol, defined by the following special axioms:

$\mathcal{W}_1: (\forall x, y, z) \left[\begin{array}{l} \text{if } x \preceq y \text{ and } y \preceq z \\ \text{then } x \preceq z \end{array} \right]$	<i>(transitivity)</i>
$\mathcal{W}_2: (\forall x, y) \left[\begin{array}{l} \text{if } x \preceq y \text{ and } y \preceq x \\ \text{then } x = y \end{array} \right]$	<i>(antisymmetry)</i>
$\mathcal{W}_3: (\forall x)[x \preceq x]$	<i>(reflexivity)</i>

As in any theory with equality, we also have the equality axioms \mathcal{E}_1 through \mathcal{E}_5 . In particular, we have two instances $\mathcal{E}_5(\preceq)$ of the *predicate-substitutivity* axiom schema $\mathcal{E}_5(q)$ in which q is taken to be the binary predicate symbol \preceq ,

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } x \preceq z \equiv y \preceq z \end{array} \right] \quad \text{(left predicate substitutivity for } \preceq \text{)}$$

and

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } z \preceq x \equiv z \preceq y \end{array} \right] \quad \text{(right predicate substitutivity for } \preceq \text{)}$$

In writing these axioms we have dropped the subscripts of z_1 and z_2 that appeared in the general schemata.

The *transitivity* axiom \mathcal{W}_1 , the *antisymmetry* axiom \mathcal{W}_2 , and the *reflexivity* axiom \mathcal{W}_3 for the weak partial ordering \preceq are independent; i.e., none of them is implied by the other two. The reader is requested to show this (in **Problem 6.8**) by constructing, for each of these axioms, a model for the theory of equality under which the axiom is false but the other two axioms are true.

The following result establishes that the equality relation of this theory can be paraphrased in terms of the weak partial ordering \preceq .

Proposition (splitting)

The sentence

$$(\forall x, y) \left[\begin{array}{c} x = y \\ \equiv \\ x \preceq y \text{ and } y \preceq x \end{array} \right] \quad (\text{splitting})$$

is valid in the theory of the weak partial ordering \preceq . \blacksquare

We give an intuitive justification.

Proof. Consider arbitrary elements x and y ; it suffices to show that

$$\begin{array}{l} \text{if } x = y \\ \text{then } x \preceq y \text{ and } y \preceq x \end{array}$$

and

$$\begin{array}{l} \text{if } x \preceq y \text{ and } y \preceq x \\ \text{then } x = y. \end{array}$$

The latter sentence follows from the *antisymmetry* axiom \mathcal{W}_2 .

To show the former sentence, suppose that

$$x = y;$$

we would like to show that

$$x \preceq y \text{ and } y \preceq x.$$

We know (by the *left predicate-substitutivity* equality axiom for \preceq , taking z to be x) that

$$\begin{array}{l} \text{if } x = y \\ \text{then } x \preceq x \equiv y \preceq x \end{array}$$

and (by the *right predicate-substitutivity* equality axiom for \preceq , taking z to be x) that

$$\begin{array}{l} \text{if } x = y \\ \text{then } x \preceq x \equiv x \preceq y. \end{array}$$

Therefore, because $x = y$ and (by the *reflexivity* axiom \mathcal{W}_3) $x \preceq x$, we have (by propositional logic)

$$x \preceq y \text{ and } y \preceq x,$$

as we wanted to show. \blacksquare

We may augment the theory of the weak partial ordering \preceq by introducing a new relation \prec , the *irreflexive restriction* of \preceq , defined by the axiom

$$\mathcal{W}_4: \quad (\forall x, y) \left[\begin{array}{c} x \prec y \\ \equiv \\ x \preceq y \text{ and not } (x = y) \end{array} \right] \quad (\text{irreflexive restriction of } \preceq)$$

Proposition (irreflexive restriction)

The irreflexive restriction \prec of \preceq is a strict partial ordering. \blacksquare

The proof is requested in **Problem 6.9**.

As in the theory of strict partial orderings, we may augment our theory of the weak partial ordering \preceq by introducing a new binary predicate symbol \succeq , denoting the *inverse* relation of \preceq . It is defined by the new special axiom:

$$\mathcal{W}_5 : (\forall x, y) [x \succeq y \equiv y \preceq x] \quad (\text{inverse})$$

Because the augmented theory is a theory with equality, we also have the corresponding instances $\mathcal{E}_5(\succeq)$ of the *predicate-substitutivity* axiom schema for equality,

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } x \succeq z \equiv y \succeq z \end{array} \right] \quad (\text{left predicate substitutivity for } \succeq)$$

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } z \succeq x \equiv z \succeq y \end{array} \right] \quad (\text{right predicate substitutivity for } \succeq)$$

We may establish a proposition that resembles the *irreflexive restriction* proposition but that applies in the other direction. Consider a new theory with equality defined by the special axioms for the theory of the strict partial ordering \prec .

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x \prec y \text{ and } y \prec z \\ \text{then } x \prec z \end{array} \right] \quad (\text{transitivity})$$

$$(\forall x) [\text{not } (x \prec x)] \quad (\text{irreflexivity})$$

Because this is a theory with equality, we also have the equality axioms.

Let us augment this theory by introducing a new *reflexive closure* relation \preceq , defined by the axiom

$$(\forall x, y) \left[\begin{array}{l} x \preceq y \\ \equiv \\ x \prec y \text{ or } x = y \end{array} \right] \quad (\text{reflexive closure of } \prec)$$

Proposition (reflexive closure)

The reflexive closure \preceq of \prec is a weak partial ordering. \blacksquare

In other words, the *transitivity*, *antisymmetry*, and *reflexivity* axioms for \preceq can be shown to be valid sentences in the new augmented theory. The proof is requested in **Problem 6.10**.

In **Problem 6.11**, the reader is asked to show the consistency and the independence of the axioms of a theory with equality.

6.7 THEORY OF GROUPS

Our second example of a theory with equality is the theory of groups. In this theory we define

- A binary function symbol $x \circ y$, denoting the *group operation*
- A constant symbol e , denoting the *identity* element
- A unary function symbol x^{-1} , denoting the *inverse* function.

Again the reader should understand that the symbols $x \circ y$, e , and x^{-1} are conventional notations for standard symbols of predicate logic, such as $f_{17}(x, y)$, a_3 , and $g_{101}(x)$.

The theory of groups is defined by the following special axioms:

$\mathcal{G}_1 :$	$(\forall x)[x \circ e = x]$	(<i>right identity</i>)
$\mathcal{G}_2 :$	$(\forall x)[x \circ x^{-1} = e]$	(<i>right inverse</i>)
$\mathcal{G}_3 :$	$(\forall x, y, z)[(x \circ y) \circ z = x \circ (y \circ z)]$	(<i>associativity</i>)

Because the theory of groups is a theory with equality, we also have the *transitivity*, *symmetry*, and *reflexivity* axioms \mathcal{E}_1 , \mathcal{E}_2 , and \mathcal{E}_3 for equality, as well as those instances of the *functional-substitutivity* axiom schema \mathcal{E}_4 that apply to the function symbols $x \circ y$ and x^{-1} of the theory:

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } x \circ z = y \circ z \end{array} \right] \quad (\text{left functional substitutivity for } \circ)$$

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x = y \\ \text{then } z \circ x = z \circ y \end{array} \right] \quad (\text{right functional substitutivity for } \circ)$$

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } x^{-1} = y^{-1} \end{array} \right] \quad (\text{functional substitutivity for inverse})$$

Because $x \circ y$ is a binary function symbol and x^{-1} is a unary function, we have two instances of the *functional-substitutivity* axiom for $x \circ y$ and one for x^{-1} .

Let us consider some models for this theory.

Examples

• The plus model

One model for the theory of groups is the *plus interpretation* \mathcal{I} over the integers, under which

- The group operation $x \circ y$ is the plus function $x_{\mathcal{I}} + y_{\mathcal{I}}$.
- The identity symbol e is the integer 0.
- The inverse function symbol x^{-1} is the unary minus function $-x_{\mathcal{I}}$.

• The times model

Another model for the theory of groups is the *times interpretation* \mathcal{J} over the positive real numbers, under which

- The group operation $x \circ y$ is the times function $x_{\mathcal{J}} \cdot y_{\mathcal{J}}$.
- The identity symbol e is the number 1.
- The inverse function x^{-1} is the reciprocal function $1/x_{\mathcal{J}}$.

The reader may confirm that each of these interpretations is a model for the theory of groups. In other words, each of the above axioms is true under both interpretations. For instance, the *right-inverse axiom*

$$(\forall x)[x \circ x^{-1} = e]$$

is true under the times interpretation because, for every positive real number r ,

$$r \cdot (1/r) = 1.$$

Note that there is no model for the theory of groups over all the real numbers under which $x \circ y$ is the times function and e is 1. Whatever unary function $g(r)$ over the reals is assigned to the inverse function symbol x^{-1} under such an interpretation, it cannot be the case that

$$0 \cdot g(0) = 1.$$

Therefore, the *right-inverse axiom* cannot be true under the interpretation. ┘

In the theory of groups, we can prove many properties from very few axioms.

Since the theory of groups is a theory with equality, we know that those sentences in our language that are valid in the theory of equality are also valid in the theory of groups. For instance, we have (by the *substitutivity-of-equality* proposition) that, for all terms s and t and any sentence $\mathcal{F}\langle s \rangle$ in the theory of groups, the universal closure of

$$\begin{array}{l} \text{if } s = t \\ \text{then } \mathcal{F}\langle s \rangle \equiv \mathcal{F}\langle t \rangle \end{array}$$

is valid in the theory of groups.

Let us show the validity of a sentence in the theory of groups.

Proposition (right cancellation)

The sentence

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x \circ z = y \circ z \\ \text{then } x = y \end{array} \right] \quad (\text{right cancellation})$$

is valid in the theory of groups. \blacksquare

Proof. Suppose that, for arbitrary elements x , y , and z ,

$$(1) \quad x \circ z = y \circ z.$$

We would like to show that $x = y$.

By (1) and the *left functional-substitutivity* equality axiom $\mathcal{E}_4(\circ)$ for the group operation \circ ,

$$(2) \quad (x \circ z) \circ z^{-1} = (y \circ z) \circ z^{-1}.$$

By the *associativity* axiom \mathcal{G}_3 for \circ ,

$$(3) \quad (x \circ z) \circ z^{-1} = x \circ (z \circ z^{-1})$$

and

$$(4) \quad (y \circ z) \circ z^{-1} = y \circ (z \circ z^{-1}).$$

By the substitutivity of equality applied to (2) and (3), we may replace $(x \circ z) \circ z^{-1}$ with $x \circ (z \circ z^{-1})$ in (2), to obtain

$$(5) \quad x \circ (z \circ z^{-1}) = (y \circ z) \circ z^{-1}.$$

Similarly, by the substitutivity of equality applied to (4) and (5), we may replace $(y \circ z) \circ z^{-1}$ with $y \circ (z \circ z^{-1})$ in (5), to obtain

$$(6) \quad x \circ (z \circ z^{-1}) = y \circ (z \circ z^{-1}).$$

By the *right-inverse* axiom \mathcal{G}_2 , we have

$$(7) \quad z \circ z^{-1} = e.$$

By the substitutivity of equality applied to (6) and (7), replacing both occurrences of $(z \circ z^{-1})$ with e in (6), we obtain

$$(8) \quad x \circ e = y \circ e.$$

By the *right-identity* axiom \mathcal{G}_1 , we have

$$(9) \quad x \circ e = x$$

and

$$(10) \quad y \circ e = y.$$

By two applications of the substitutivity of equality applied to (8), (9), and (10), replacing $x \circ e$ with x in (8) and replacing $y \circ e$ with y in the result, we obtain

$$(11) \quad x = y.$$

This is the desired conclusion. \blacksquare

Some other valid sentences of the theory of groups are

$$(\forall x)[e \circ x = x] \quad (\text{left identity})$$

$$(\forall x)[x^{-1} \circ x = e] \quad (\text{left inverse})$$

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } z \circ x = z \circ y \\ \text{then } x = y \end{array} \right] \quad (\text{left cancellation})$$

$$(\forall x) \left[\begin{array}{l} \text{if } x \circ x = x \\ \text{then } x = e \end{array} \right] \quad (\text{nonidempotence})$$

The proofs of the validity of these properties in the theory of groups are left as an exercise (**Problem 6.12**).

Once we have proved these properties for groups, we know that they are true under all models for groups. For example, because the *nonidempotence* property above is valid in the theory of groups and because the *plus* interpretation over the integers is a model for the theory, we can conclude that

for every integer x ,
 if $x + x = x$,
 then $x = 0$.

Similarly, because the *times* interpretation over the positive real numbers is a model for the theory of groups, we can conclude that

for every positive real number x ,
 if $x \cdot x = x$,
 then $x = 1$.

COMMUTATIVITY

Not every property of plus and times is valid in the theory of groups. For example, even though plus is commutative, that is,

$$x + y = y + x$$

is true for all integers x and y , and times is also commutative, that is,

$$x \cdot y = y \cdot x$$

is true for all positive real numbers, the group operation \circ is not necessarily commutative, i.e., the corresponding sentence

$$(\forall x, y)[x \circ y = y \circ x]$$

is not valid in the theory of groups. To see this, it suffices to find a single model for the theory under which the commutativity sentence is not true.

Example (permutation model). Consider the set Π of all permutations on the set of three elements $S = \{A, B, C\}$. These are the unary functions that map distinct elements of S into distinct elements of S ; there are precisely six of them:

- The *identity* π_0 , which leaves all elements fixed; that is,

$$\pi_0(A) = A \quad \pi_0(B) = B \quad \pi_0(C) = C.$$

- The *transpositions* π_A, π_B , and π_C , which leave one element fixed but interchange the other two; that is,

$$\pi_A(A) = A \quad \pi_A(B) = C \quad \pi_A(C) = B$$

$$\pi_B(A) = C \quad \pi_B(B) = B \quad \pi_B(C) = A$$

$$\pi_C(A) = B \quad \pi_C(B) = A \quad \pi_C(C) = C.$$

- The *cycles* π_+ and π_- , which alter all the elements; that is,

$$\pi_+(A) = B \quad \pi_+(B) = C \quad \pi_+(C) = A$$

$$\pi_-(A) = C \quad \pi_-(B) = A \quad \pi_-(C) = B.$$

For all permutations π and π' , let the *composition permutation* $\pi \otimes \pi'$ be the permutation obtained by applying first π and then π' ; in other words, for any element s of S ,

$$(\pi \otimes \pi')(s) = \pi'(\pi(s)).$$

The *composition function* maps any two permutations π and π' into their composition permutation $\pi \otimes \pi'$. For example,

$$[\pi_A \otimes \pi_C](A) = \pi_C(\pi_A(A)) = \pi_C(A) = B$$

$$[\pi_A \otimes \pi_C](B) = \pi_C(\pi_A(B)) = \pi_C(C) = C$$

$$[\pi_A \otimes \pi_C](C) = \pi_C(\pi_A(C)) = \pi_C(B) = A.$$

Note that, for each element s of S , $[\pi_A \otimes \pi_C](s) = \pi_+(s)$; thus $\pi_A \otimes \pi_C = \pi_+$.

For any permutation π , let the *inverse permutation* $\tilde{\pi}$ be defined so that, for any elements s and s' of S ,

$$\pi(s) = s' \text{ if and only if } \tilde{\pi}(s') = s.$$

The *inverse function* maps any permutation π into its inverse permutation $\tilde{\pi}$. For example,

$$\text{since } \pi_+(A) = B, \text{ we have } \tilde{\pi}_+(B) = A;$$

$$\text{since } \pi_+(B) = C, \text{ we have } \tilde{\pi}_+(C) = B;$$

$$\text{since } \pi_+(C) = A, \text{ we have } \tilde{\pi}_+(A) = C.$$

Note that, for each element s of S , $\tilde{\pi}_+(s) = \pi_-(s)$; thus $\tilde{\pi}_+ = \pi_-$.

Now consider the *permutation interpretation* \mathcal{K} , whose domain is the set of permutations of elements of S , under which

- The function symbol $x \circ y$ is the composition function $x_{\mathcal{K}} \otimes y_{\mathcal{K}}$.
- The constant e is the identity permutation π_0 .
- The function symbol x^{-1} is the inverse function $\widetilde{x_{\mathcal{K}}}$.

In **Problem 6.13**, the reader is requested to confirm that \mathcal{K} is a model for the theory of groups, i.e., that the *right-identity* axiom \mathcal{G}_1 , the *right-inverse* axiom \mathcal{G}_2 , and the *associativity* axiom \mathcal{G}_3 are true under \mathcal{K} .

On the other hand, the *commutativity* property

$$(\forall x, y)[x \circ y = y \circ x]$$

is not true under \mathcal{K} . For we have already observed that

$$\pi_A \otimes \pi_C = \pi_+.$$

On the other hand,

$$[\pi_C \otimes \pi_A](A) = \pi_A(\pi_C(A)) = \pi_A(B) = C$$

$$[\pi_C \otimes \pi_A](B) = \pi_A(\pi_C(B)) = \pi_A(A) = A$$

$$[\pi_C \otimes \pi_A](C) = \pi_A(\pi_C(C)) = \pi_A(C) = B.$$

Thus, for each element s of S , $[\pi_C \otimes \pi_A](s) = \pi_-(s)$; that is,

$$\pi_C \otimes \pi_A = \pi_-.$$

Because $\pi_+ \neq \pi_-$, we have

$$\pi_A \otimes \pi_C \neq \pi_C \otimes \pi_A,$$

showing that the composition function on permutations of S is not commutative. \blacksquare

Because we have found a model for the theory of groups under which the *commutativity* property is not true, we have shown that the property is not valid in the theory. If we wish to consider only those models under which the group operation $x \circ y$ is commutative, we can augment the theory of groups by adding the new axiom

$\mathcal{G}_4: \quad (\forall x, y)[x \circ y = y \circ x]$	<i>(commutativity)</i>
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The new theory is called the theory of *commutative* (or *abelian*) groups. All the valid sentences of the original theory of groups are also valid in this augmented theory. For the theory of commutative groups, the *plus* and *times* interpretations are still models but the permutation interpretation \mathcal{K} is not.

In **Problem 6.14**, the reader is asked to prove the validity of a sentence in the theory of monoids.

6.8 THEORY OF PAIRS

Another example of a theory with equality is the theory of pairs. Intuitively speaking, in this theory we have certain basic elements, called *atoms*, from which we construct pairs of form $\langle x_1, x_2 \rangle$, where each component x_1 and x_2 is an atom. For example, if the atoms are A, B, and C, then

$$\langle A, A \rangle, \quad \langle A, B \rangle, \quad \langle B, A \rangle, \quad \text{and} \quad \langle C, A \rangle$$

are distinct pairs. The intended domain consists of both the atoms and the pairs of atoms.

In the theory of pairs, we define

- A unary predicate symbol $atom(x)$
- A unary predicate symbol $pair(x)$
- A binary function symbol $\langle x_1, x_2 \rangle$, denoting the *pairing* function.

The predicate symbol $atom(x)$ is true if x is an atom and false if x is a pair; $pair(x)$ is true if x is a pair and false if x is an atom. The value of the pairing function $\langle x_1, x_2 \rangle$ is the pair whose first element is the atom x_1 and whose second element is the atom x_2 .

Again, do not be confused; we are not adding a new notation $\langle x_1, x_2 \rangle$ to the formal language of predicate logic; we are merely adopting informally the familiar mathematical notation for a pair to represent a standard predicate-logic binary function symbol, such as $f_{101}(x_1, x_2)$.

The theory of pairs is defined by the following special axioms:

$$\mathcal{P}_1: \quad (\forall x) \left[\begin{array}{c} pair(x) \\ \equiv \\ (\exists x_1, x_2) \left[\begin{array}{c} atom(x_1) \text{ and } atom(x_2) \\ \text{and} \\ x = \langle x_1, x_2 \rangle \end{array} \right] \end{array} \right] \quad (pair)$$

In other words, every pair is of form $\langle x_1, x_2 \rangle$, where x_1 and x_2 are atoms.

$$\mathcal{P}_2: \quad (\forall x) [not (atom(x) \text{ and } pair(x))] \quad (disjoint)$$

In other words, no domain element is both an atom and a pair.

$$\mathcal{P}_3: \quad (\forall x_1, x_2, y_1, y_2) \left[\begin{array}{c} \text{if} \left[\begin{array}{c} atom(x_1) \text{ and } atom(x_2) \\ \text{and} \\ atom(y_1) \text{ and } atom(y_2) \end{array} \right] \\ \text{then if } \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \\ \text{then } x_1 = y_1 \text{ and } x_2 = y_2 \end{array} \right] \quad (uniqueness)$$

In other words, a pair can be constructed in only one way from two atoms.

Remark (pairs of nonatoms). Note that the axioms do not specify the result of applying the pairing function $\langle x_1, x_2 \rangle$ if x_1 or x_2 is itself a pair rather than an atom. Although expressions of this form are legal in the language and although they must have some values under any interpretation, the axioms do not determine these values. Thus if x_1 and x_2 are not both atoms under a given model, the term $\langle x_1, x_2 \rangle$ might have the value A , $\langle A, B \rangle$, $\langle B, C \rangle$, or any other domain element. We simply do not care what the value of the pairing function is in this case. \perp

Because the theory of pairs is a theory with equality, we also have the equality axioms \mathcal{E}_1 through \mathcal{E}_5 , including the appropriate instances of the *functional-substitutivity* axiom schema \mathcal{E}_4 :

$$(\forall x_1, x'_1, x_2) \left[\begin{array}{l} \text{if } x_1 = x'_1 \\ \text{then } \langle x_1, x_2 \rangle = \langle x'_1, x_2 \rangle \end{array} \right]$$

(left functional substitutivity for pairing)

$$(\forall x_1, x_2, x'_2) \left[\begin{array}{l} \text{if } x_2 = x'_2 \\ \text{then } \langle x_1, x_2 \rangle = \langle x_1, x'_2 \rangle \end{array} \right]$$

(right functional substitutivity for pairing)

We also have the instances of the *predicate-substitutivity* axiom schema \mathcal{E}_5 for equality that apply to the *atom* predicate symbol,

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } \text{atom}(x) \equiv \text{atom}(y) \end{array} \right]$$

(predicate substitutivity for atom)

and to the *pair* predicate symbol,

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } \text{pair}(x) \equiv \text{pair}(y) \end{array} \right] \quad (\text{predicate substitutivity for pair})$$

because these symbols are in our vocabulary.

Example (pairs of integers). Consider the interpretation \mathcal{I} over the set of integers and pairs of integers under which

- The unary predicate symbol $\text{atom}(x)$ is the relation that is true if $x_{\mathcal{I}}$ is an integer and false if $x_{\mathcal{I}}$ is a pair.
- The unary predicate symbol $\text{pair}(x)$ is the relation that is true if $x_{\mathcal{I}}$ is a pair and false if $x_{\mathcal{I}}$ is an integer.
- The binary function symbol $\langle x_1, x_2 \rangle$ is any function k such that $k(d_1, d_2)$ is the pair $\langle d_1, d_2 \rangle$, for all integers d_1 and d_2 ; we do not care what the value of $k(d_1, d_2)$ is if d_1 or d_2 is itself a pair.

The reader may confirm that \mathcal{I} is a model for the theory of pairs. \perp

THE FIRST AND SECOND FUNCTIONS

Let us now augment our theory of pairs by introducing two unary function symbols *first* and *second*. Intuitively speaking, *first*(*x*) and *second*(*x*) are the first and second elements, respectively, of the pair *x*. The axioms that define these functions follow:

$$\begin{array}{ll} \mathcal{P}_4 : & (\forall x_1, x_2) \left[\begin{array}{l} \text{if } \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{then } \text{first}(\langle x_1, x_2 \rangle) = x_1 \end{array} \right] \quad (\text{first}) \\ \mathcal{P}_5 : & (\forall x_1, x_2) \left[\begin{array}{l} \text{if } \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{then } \text{second}(\langle x_1, x_2 \rangle) = x_2 \end{array} \right] \quad (\text{second}) \end{array}$$

Note that the axioms do not specify the value of an expression of the form *first*(*x*) or *second*(*x*) if *x* is an atom rather than a pair. We do not care what value is assigned to such an expression under a model for the augmented theory.

Because the augmented theory is a theory with equality, we have the appropriate instances of the *functional-substitutivity* axiom schema \mathcal{E}_4 for equality,

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } \text{first}(x) = \text{first}(y) \end{array} \right] \quad (\text{functional substitutivity for first})$$

$$(\forall x, y) \left[\begin{array}{l} \text{if } x = y \\ \text{then } \text{second}(x) = \text{second}(y) \end{array} \right] \quad (\text{functional substitutivity for second})$$

We can easily establish the validity of the following sentences in the theory of pairs: For the *first* function we have

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } \text{atom}(\text{first}(x)) \end{array} \right] \quad (\text{sort of first})$$

For the *second* function we have

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } \text{atom}(\text{second}(x)) \end{array} \right] \quad (\text{sort of second})$$

Often we refer to a unary predicate symbol, which characterizes a set of domain elements, as a “sort.” The above properties are called the *sort* properties of the *first* and the *second* function, respectively, because they establish that if a given element *x* is of the sort *pair*, then *first*(*x*) and *second*(*x*) are elements of the sort *atom*.

THE DECOMPOSITION PROPERTY

In the augmented theory of pairs we can establish the following result.

Proposition (decomposition)

The sentence

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } x = \langle \text{first}(x), \text{second}(x) \rangle \end{array} \right] \quad (\text{decomposition})$$

is valid in the augmented theory of pairs. \blacksquare

In other words, any pair is the result of pairing its first and second elements.

Proof. For an arbitrary element x , suppose that
 $\text{pair}(x)$.

We would like to show that

$$x = \langle \text{first}(x), \text{second}(x) \rangle.$$

We know (by the *pair* axiom \mathcal{P}_1) that

$$(\exists x_1, x_2) \left[\begin{array}{l} \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{and} \\ x = \langle x_1, x_2 \rangle \end{array} \right].$$

Let x_1 and x_2 be elements such that

$$\begin{array}{l} \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{and} \\ x = \langle x_1, x_2 \rangle. \end{array}$$

Then (by the *symmetry* axiom for equality)

$$\langle x_1, x_2 \rangle = x.$$

We have (by the definitions of *first* and *second*)

$$\begin{array}{l} \text{if } \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{then } \text{first}(\langle x_1, x_2 \rangle) = x_1 \end{array}$$

and

$$\begin{array}{l} \text{if } \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{then } \text{second}(\langle x_1, x_2 \rangle) = x_2. \end{array}$$

Therefore (by propositional logic, because $\text{atom}(x_1)$ and $\text{atom}(x_2)$)

$$\text{first}(\langle x_1, x_2 \rangle) = x_1 \quad \text{and} \quad \text{second}(\langle x_1, x_2 \rangle) = x_2,$$

that is (by the substitutivity of equality, because $\langle x_1, x_2 \rangle = x$),

$$\text{first}(x) = x_1 \quad \text{and} \quad \text{second}(x) = x_2.$$

Therefore (by the *symmetry* axiom for equality)

$$x_1 = \text{first}(x) \quad \text{and} \quad x_2 = \text{second}(x).$$

Finally (by two applications of the substitutivity of equality, because $x = \langle x_1, x_2 \rangle$), we have

$$x = \langle \text{first}(x), \text{second}(x) \rangle,$$

as we wanted to show. \blacksquare

6.9 RELATIVIZED QUANTIFIERS

We now introduce a notational convention, that of *relativized quantifiers*, to abbreviate sentences in predicate logic. This convention has the effect of allowing quantifiers to range over particular subsets of the domain instead of over the entire domain.

Definition (relativized quantifier)

For any unary predicate symbol p and sentence \mathcal{F} ,

$$(\forall p x)\mathcal{F} \quad \text{stands for} \quad (\forall x) \left[\begin{array}{l} \text{if } p(x) \\ \text{then } \mathcal{F} \end{array} \right]$$

$$(\exists p x)\mathcal{F} \quad \text{stands for} \quad (\exists x) [p(x) \text{ and } \mathcal{F}]. \quad \text{J}$$

Examples. In the theory of pairs, for a binary predicate symbol q , the sentence

$$(\forall \text{ atom } x_1)[\text{not } q(x_1, x_1)]$$

stands for

$$(\forall x_1) \left[\begin{array}{l} \text{if } \text{atom}(x_1) \\ \text{then } \text{not } q(x_1, x_1) \end{array} \right].$$

The sentence

$$(\forall \text{ pair } x)(\exists \text{ atom } x_1)[\text{first}(x) = x_1]$$

stands for

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } (\exists \text{ atom } x_1)[\text{first}(x) = x_1] \end{array} \right],$$

which stands for

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } (\exists x_1) \left[\begin{array}{l} \text{atom}(x_1) \\ \text{and} \\ \text{first}(x) = x_1 \end{array} \right] \end{array} \right]. \quad \text{J}$$

We can apply the abbreviation to sequences of relativized quantifiers.

Definition (multiple relativized quantifiers)

For any unary predicate symbols p_1, p_2, \dots, p_n and sentence \mathcal{F} ,

$$(\forall p_1 x_1)(\forall p_2 x_2) \dots (\forall p_n x_n) \mathcal{F}$$

stands for

$$(\forall x_1, x_2, \dots, x_n) \left[\begin{array}{l} \text{if } p_1(x_1) \text{ and } p_2(x_2) \text{ and } \dots \text{ and } p_n(x_n) \\ \text{then } \mathcal{F} \end{array} \right],$$

and

$$(\exists p_1 x_1) (\exists p_2 x_2) \dots (\exists p_n x_n) \mathcal{F}$$

stands for

$$(\exists x_1, x_2, \dots, x_n) \left[\begin{array}{c} p_1(x_1) \text{ and } p_2(x_2) \text{ and } \dots \text{ and } p_n(x_n) \\ \text{and} \\ \mathcal{F} \end{array} \right].$$

As a special case,

$$(\forall p x_1, x_2, \dots, x_n) \mathcal{F}$$

stands for

$$(\forall p x_1) (\forall p x_2) \dots (\forall p x_n) \mathcal{F},$$

which stands for

$$(\forall x_1, x_2, \dots, x_n) \left[\begin{array}{c} \text{if } p(x_1) \text{ and } p(x_2) \text{ and } \dots \text{ and } p(x_n) \\ \text{then } \mathcal{F} \end{array} \right].$$

Similarly,

$$(\exists p x_1, x_2, \dots, x_n) \mathcal{F}$$

stands for

$$(\exists x_1, x_2, \dots, x_n) \left[\begin{array}{c} p(x_1) \text{ and } p(x_2) \text{ and } \dots \text{ and } p(x_n) \\ \text{and} \\ \mathcal{F} \end{array} \right].$$

Examples. In the theory of pairs, the sentence

$$(\forall \text{atom } x_1, x_2) [\text{first}(\langle x_1, x_2 \rangle) = x_1]$$

stands for

$$(\forall x_1, x_2) \left[\begin{array}{c} \text{if } \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{then } \text{first}(\langle x_1, x_2 \rangle) = x_1 \end{array} \right].$$

The sentence

$$(\exists \text{pair } x, y) [x \preceq y \text{ and } y \preceq x]$$

stands for

$$(\exists x, y) \left[\begin{array}{c} \text{pair}(x) \text{ and } \text{pair}(y) \\ \text{and} \\ x \preceq y \text{ and } y \preceq x \end{array} \right].$$

The sentence

$$(\forall \text{pair } x) (\exists \text{atom } x_1, x_2) [x = \langle x_1, x_2 \rangle]$$

stands for

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } (\exists \text{ atom } x_1, x_2) [x = \langle x_1, x_2 \rangle] \end{array} \right],$$

which stands for

$$(\forall x) \left[\begin{array}{l} \text{if } \text{pair}(x) \\ \text{then } (\exists x_1, x_2) \left[\begin{array}{l} \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{and} \\ x = \langle x_1, x_2 \rangle \end{array} \right] \end{array} \right].$$

The relativized quantifier notation can make the sentences in our theory of pairs somewhat clearer. For example, the definition \mathcal{P}_1 of the *pair* relation, which was originally written as

$$(\forall x) \left[\begin{array}{l} \text{pair}(x) \\ \equiv \\ (\exists x_1, x_2) \left[\begin{array}{l} \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{and} \\ x = \langle x_1, x_2 \rangle \end{array} \right] \end{array} \right],$$

can now be abbreviated as

$$(\forall x) \left[\begin{array}{l} \text{pair}(x) \\ \equiv \\ (\exists \text{ atom } x_1, x_2) [x = \langle x_1, x_2 \rangle] \end{array} \right].$$

The *uniqueness* axiom \mathcal{P}_3 for pairs, which was originally written as

$$(\forall x_1, x_2, y_1, y_2) \left[\begin{array}{l} \text{if } \left[\begin{array}{l} \text{atom}(x_1) \text{ and } \text{atom}(x_2) \\ \text{and} \\ \text{atom}(y_1) \text{ and } \text{atom}(y_2) \end{array} \right] \\ \text{then if } \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \\ \text{then } x_1 = y_1 \text{ and } x_2 = y_2 \end{array} \right],$$

can now be abbreviated as

$$(\forall \text{ atom } x_1, x_2, y_1, y_2) \left[\begin{array}{l} \text{if } \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \\ \text{then } x_1 = y_1 \text{ and } x_2 = y_2 \end{array} \right].$$

The definitions \mathcal{P}_4 and \mathcal{P}_5 of the *first* and *second* functions can now be abbreviated as

$$(\forall \text{ atom } x_1, x_2) [\text{first}(\langle x_1, x_2 \rangle) = x_1]$$

and

$$(\forall \text{ atom } x_1, x_2) [\text{second}(\langle x_1, x_2 \rangle) = x_2].$$

The *decomposition* proposition may be written as

$$(\forall \text{ pair } x) [x = \langle \text{first}(x), \text{second}(x) \rangle].$$

When we need to prove a sentence expressed in terms of relativized quantifiers, we can always abandon the abbreviation, rephrase the sentence in terms of ordinary quantifiers, and use ordinary predicate logic arguments. Alternatively, we can identify valid sentence schemata with relativized quantifiers and retain the abbreviation. The relativized-quantifier schemata resemble some of the ordinary valid sentence schemata.

In particular, for all unary predicate symbols p and q , we can establish the validity of the universal closures of the following sentence schemata:

• Reversal of quantifiers

$$(\forall p x)(\forall q y) \mathcal{F}$$

$$\equiv$$

$$(\forall q y)(\forall p x) \mathcal{F}$$

$$(\exists p x)(\exists q y) \mathcal{F}$$

$$\equiv$$

$$(\exists q y)(\exists p x) \mathcal{F}$$

• Duality of quantifiers

$$(\forall p x)[\text{not } \mathcal{F}]$$

$$\equiv$$

$$\text{not } (\exists p x) \mathcal{F}$$

$$(\exists p x)[\text{not } \mathcal{F}]$$

$$\equiv$$

$$\text{not } (\forall p x) \mathcal{F}.$$

Let us justify the last of these equivalences.

Proposition (duality of relativized quantifiers)

For any unary predicate symbol p ,

$$(\exists p x)[\text{not } \mathcal{F}] \equiv \text{not } (\forall p x) \mathcal{F}. \quad \blacksquare$$

Proof. We have that

$$(\exists p x)[\text{not } \mathcal{F}]$$

is an abbreviation of

$$(\exists x) [p(x) \text{ and } \text{not } \mathcal{F}],$$

which is equivalent (by propositional logic) to

$$(\exists x) \text{ not } \left[\begin{array}{l} \text{if } p(x) \\ \text{then } \mathcal{F} \end{array} \right],$$

which is equivalent (by the duality property of ordinary quantifiers) to

$$\text{not } (\forall x) \left[\begin{array}{l} \text{if } p(x) \\ \text{then } \mathcal{F} \end{array} \right],$$

which may be abbreviated as

$$\text{not } (\forall p x) \mathcal{F}.$$

This establishes the desired equivalence. \blacksquare

The reader is requested (in **Problem 6.15**) to prove two additional equivalences concerning relativized quantifiers.

Remark (pitfall). We must be careful not to apply properties of ordinary quantifiers blindly to relativized quantifiers. For example, the sentence

$$\begin{array}{l} \text{if } (\forall x)q(x) \\ \text{then } (\exists x)q(x) \end{array}$$

is valid. However, for any unary predicate symbol p , the analogous sentence with relativized quantifiers,

$$\begin{array}{l} \text{if } (\forall p x)q(x) \\ \text{then } (\exists p x)q(x), \end{array}$$

is not valid. This sentence stands for

$$\begin{array}{l} \text{if } (\forall x)[\text{if } p(x) \text{ then } q(x)] \\ \text{then } (\exists x)[p(x) \text{ and } q(x)], \end{array}$$

which is false under any interpretation under which $(\exists x)p(x)$ is false. Under such an interpretation, the antecedent of this implication is true vacuously, but its consequent is false. On the other hand, the sentence is true under any interpretation under which $(\exists x)p(x)$ is true. \perp

In **Problem 6.16**, the reader is asked to define a theory of triples analogous to the theory of pairs.

PROBLEMS

Problem 6.1 (family theory) page 308

In the family theory show that

$$\begin{array}{l} \text{if Alice is the parent of her own father,} \\ \text{then Alice's father is his own grandfather.} \end{array}$$

(First express the sentence in predicate logic, then give an intuitive argument to show it is valid in the family theory. Let the constant a denote Alice.)

Problem 6.2 (strict partial ordering) page 313

Construct an interpretation over the finite domain $\{A, B, C\}$ under which

- (a) The *transitivity* axiom S_1 is true but the *irreflexivity* axiom S_2 is false.
- (b) The *irreflexivity* axiom S_2 is true but the *transitivity* axiom S_1 is false.

Problem 6.3 (equivalence relation) page 317

Consider the following “proof” that, in the theory of the equivalence relation \approx , the *transitivity* axiom Q_1 and the *symmetry* axiom Q_2 imply the *reflexivity* axiom Q_3 .

We would like to show that, for an arbitrary element x , $x \approx x$. Let y be any element such that $x \approx y$. Then (by the *symmetry* axiom Q_2) we have $y \approx x$. Therefore (by the *transitivity* axiom Q_1 , because $x \approx y$ and $y \approx x$) we have $x \approx x$, as we wanted to show.

We have already shown, however, that the three axioms for the theory of equivalence are independent, i.e., that no two of them imply the third. Find the fallacious step in the above argument.

***Problem 6.4 (nonvalid equality sentence)** page 322

Show that the following sentence is not valid in the theory of equality:

$$(\exists x)(\forall y)p(x, y)$$

or

$$\text{not } (\exists x)(\forall y)[\text{if not } (x = y) \text{ then } p(x, y)]$$

or

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } p(x, y) \text{ and } p(y, y) \text{ and } p(y, z) \\ \text{then } p(x, z) \end{array} \right].$$

Hint: Construct an interpretation over a finite domain under which this sentence is false.

Problem 6.5 (replacement) page 324

Show that the *replacement* proposition would not hold if we had applied a partial substitution rather than a total substitution. More precisely,

(a) *Universal*

Find a term t and a sentence $\mathcal{F}\langle x \rangle$ such that

$$(\forall x)[\text{if } x = t \text{ then } \mathcal{F}\langle x \rangle]$$

is not equivalent to $\mathcal{F}\langle t \rangle$.

(b) *Existential*

Find a term t and a sentence $\mathcal{F}\langle x \rangle$ such that

$$(\exists x)[x = t \text{ and } \mathcal{F}\langle x \rangle]$$

is not equivalent to $\mathcal{F}\langle t \rangle$.

Show that the *replacement* proposition would not hold if x were allowed to occur free in t . More precisely,

(c) *Universal*

Find a term t and a sentence $\mathcal{F}[x]$ such that x occurs free in t and the sentence

$$(\forall x)[\text{if } x = t \text{ then } \mathcal{F}[x]]$$

is not equivalent to $\mathcal{F}[t]$.

(d) *Existential*

Find a term t and a sentence $\mathcal{F}[x]$ such that x occurs free in t and the sentence

$$(\exists x)[x = t \text{ and } \mathcal{F}[x]]$$

is not equivalent to $\mathcal{F}[t]$.

Justify your answer in each case.

Problem 6.6 (conditional terms) page 324

Establish the validity of the following sentences in the theory of equality:

(a) *True*

$$(\text{if true then } a \text{ else } b) = a$$

(b) *False*

$$(\text{if false then } a \text{ else } b) = b$$

(c) *Distributivity*

$$(\forall x) \left[\begin{array}{l} f(\text{if } p(x) \text{ then } a \text{ else } b) \\ = \\ \text{if } p(x) \text{ then } f(a) \text{ else } f(b) \end{array} \right].$$

Problem 6.7 (uniqueness of equality) page 324

Prove the *uniqueness-of-equality* proposition.

Problem 6.8 (weak partial ordering) page 325

Prove that the *transitivity* axiom \mathcal{W}_1 , the *antisymmetry* axiom \mathcal{W}_2 , and the *reflexivity* axiom \mathcal{W}_3 for the theory of a weak partial ordering \preceq are independent; i.e., for each of these axioms, there is a model for the theory of equality under which the axiom is false but the other two axioms are true.

Problem 6.9 (irreflexive restriction) page 327

Prove the *irreflexive restriction* proposition.

Problem 6.10 (reflexive closure) page 328

Prove the *reflexive closure* proposition.

Problem 6.11 (consistency and independence) page 328

Consider the theory with equality defined by the following special axioms:

$$(\forall x)[x = f(f(f(x)))] \quad (\text{three})$$

$$(\forall x, y) \left[\begin{array}{l} \text{if } x \neq y \\ \text{then } x = f(y) \text{ or } y = f(x) \end{array} \right] \quad (\text{connected})$$

$$(\forall x)[\text{if } p(x) \text{ then not } p(f(x))] \quad (\text{skip})$$

$$(\exists x)p(x) \quad (\text{one})$$

- (a) Show that the theory is consistent.

Hint: Construct a model with a domain of exactly three elements.

- (b) Show that the axioms are independent.

Problem 6.12 (theory of groups) page 331

Prove informally the validity of the following properties in the theory of groups:

- (a) *Left identity*

$$(\forall x)[e \circ x = x]$$

[*Hint:* For an arbitrary element x , show that $(e \circ x) \circ x^{-1} = x \circ x^{-1}$.]

- (b) *Left inverse*

$$(\forall x)[x^{-1} \circ x = e]$$

[*Hint:* For an arbitrary element x , show that $(x^{-1} \circ x) \circ x^{-1} = e \circ x^{-1}$.]

- (c) *Left cancellation*

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } z \circ x = z \circ y \\ \text{then } x = y \end{array} \right]$$

[*Hint:* For arbitrary elements x , y , and z such that $z \circ x = z \circ y$, show that $(z^{-1} \circ z) \circ x = (z^{-1} \circ z) \circ y$.]

- (d) *Nonidempotence*

$$(\forall x) \left[\begin{array}{l} \text{if } x \circ x = x \\ \text{then } x = e \end{array} \right]$$

Note: The order in which these sentences are presented is significant; the proof of each may rely on the validity of the previous sentences.

Problem 6.13 (permutation interpretation) page 333

Show that the permutation interpretation \mathcal{K} is a model for the theory of groups; i.e., show that

- (a) The *right-identity* axiom \mathcal{G}_1
 - (b) The *right-inverse* axiom \mathcal{G}_2
 - (c) The *associativity* axiom \mathcal{G}_3
- are true under \mathcal{K} .

Problem 6.14 (theory of monoids) page 333

Consider the theory of monoids, a theory with equality defined by the following special axioms:

$$\begin{array}{ll}
 (\forall x)[x \circ e = x] & \text{(right identity)} \\
 (\forall x)[e \circ x = x] & \text{(left identity)} \\
 (\forall x, y, z)[(x \circ y) \circ z = x \circ (y \circ z)] & \text{(associativity)}
 \end{array}$$

- (a) Determine whether the above axioms are independent. Justify your answer.
- (b) Prove informally the validity of the following sentence in the theory of monoids:

$$(\forall x, y, z) \left[\begin{array}{l} \text{if } x \circ y = e \text{ and } y \circ z = e \\ \text{then } x = z \end{array} \right].$$

Problem 6.15 (relativized quantifiers) page 342

For all unary predicate symbols p , establish the validity of the universal closures of the following sentence schemata:

- | | |
|--|--|
| <p>(a) <i>Reversal</i></p> $ \begin{array}{l} (\exists p x)(\exists q y)\mathcal{F} \\ \equiv \\ (\exists q y)(\exists p x)\mathcal{F} \end{array} $ | <p>(b) <i>Distributivity</i></p> $ \begin{array}{l} (\forall p x)[\mathcal{F} \text{ and } \mathcal{G}] \\ \equiv \\ (\forall p x)\mathcal{F} \text{ and } (\forall p x)\mathcal{G}. \end{array} $ |
|--|--|

Problem 6.16 (theory of triples) page 342

Define a theory of triples, analogous to the theory of pairs, in which, intuitively speaking, $\langle x_1, x_2, x_3 \rangle$ is a triple of three atoms x_1 , x_2 , and x_3 . Provide the basic axioms for this theory. Use the relativized-quantifier notation.