7

Deductive Tableaux

We have introduced a deductive-tableau system to prove the validity of sentences in predicate logic. In this chapter, we adapt the system to prove validity in axiomatic theories.

7.1 FINITE THEORIES

In our discussion of axiomatic theories in the previous chapter, we showed how to describe a particular theory by presenting a (possibly infinite) set of closed sentences,

$$\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3,\ \ldots,$$

which are the axioms of the theory. We defined an interpretation to be a *model* of the theory if each axiom $\mathcal{A}_i$ of the theory is true under the interpretation. A closed sentence $\mathcal{S}$ of the theory is *valid* in the theory if $\mathcal{S}$ is true under every model for the theory. Similarly, two sentences are *equivalent* in the theory if they have the same truth-value under every model for the theory.

For instance, we defined the theory of strict partial orderings by the two axioms

$(\forall x, y, z)\begin{bmatrix} \textit{if}\ \ x \prec y\ \ \textit{and}\ \ y \prec z \\ \textit{then}\ \ x \prec z \end{bmatrix}$	(*transitivity*)
$(\forall x)[\textit{not}\ (x \prec x)]$	(*irreflexivity*)

Here we take $\prec$ to stand for any binary predicate symbol. This theory has many models, including the less-than ordering $<$ over the integers and the proper-subset relation $\subset$ over the sets. To determine that a closed sentence $\mathcal{S}$ is valid in the strict partial-ordering theory, we must establish that $\mathcal{S}$ is true under all these models.

A tableau is said to be *valid in a theory* if its associated sentence is valid in the theory, and two tableaux are said to be *equivalent in a theory* if their associated sentences are equivalent in the theory.

Suppose we wish to prove that a closed sentence $\mathcal{S}$ is valid in a *finite theory*, that is, one defined by a finite set of axioms. In the deductive-tableau framework, we can do this by proving in predicate logic the initial tableau

assertions	goals
$\mathcal{A}_1$	
$\mathcal{A}_2$	
$\vdots$	
$\mathcal{A}_n$	
	$\mathcal{S}$

where each assertion $\mathcal{A}_i$ is a sentence known to be valid in the theory, either because it is an axiom or because it has been previously proved valid in the theory. A sentence that has been proved valid in a theory is called a *theorem of the theory.*

For example, to establish that a closed sentence $\mathcal{S}$ is valid in the theory of strict partial orderings, defined by the *transitivity* and *irreflexivity* axioms, we prove in predicate logic the initial tableau

assertions		goals
if $x \prec y$ *and* $y \prec z$ *then* $x \prec z$	(*transitivity*)	
not $(x \prec x)$	(*irreflexivity*)	
		$\mathcal{S}$

Here, by outermost skolemization (Section 5.9), we have dropped the outermost universal quantifiers from the two assertions.

Once we have proved the validity of a sentence $\mathcal{S}$ in the theory of strict partial orderings, we may add $\mathcal{S}$ as an assertion in any future tableaux.

Example (theory of strict partial orderings). Suppose we would like to show that the *asymmetry* property is valid in the theory of the strict partial ordering $\prec$; that is,

$$(\forall x, y)\begin{bmatrix} \text{if } x \prec y \\ \text{then not } (y \prec x) \end{bmatrix} \qquad (asymmetry)$$

is valid. This was established informally in Section 6.3.

For this purpose it suffices to prove in predicate logic the tableau

assertions	goals
if $x \prec y$ *and* $y \prec z$ *then* $x \prec z$ (*transitivity*)	
not $(x \prec x)$ (*irreflexivity*)	
	G1. $(\forall x, y)^{\forall}\begin{bmatrix} \textit{if } x \prec y \\ \textit{then not } (y \prec x) \end{bmatrix}$

Note that we did not number the two axioms (as A1 and A2). We shall usually refer to such assertions (axioms or theorems) by name.

Applying the $\forall$-*elimination* rule twice in succession to goal G1, replacing the bound variables x and y with the skolem constants a and b, respectively, we obtain the goal

	G2. *if* $a \prec b$ *then not* $(b \prec a)$

By the *if-split* rule, this decomposes into

A3. $\boxed{a \prec b}^{-}$	
	G4. *not* $\boxed{b \prec a}^{-}$

By the *resolution* rule, applied to assertion A3 and the *transitivity* axiom

if $\boxed{x \prec y}^{+}$ *and* $y \prec z$ *then* $x \prec z$	

with $\{x \leftarrow a,\ y \leftarrow b\}$, we obtain

A5. *if* $\boxed{b \prec z}^{+}$ *then* $a \prec z$	

By the *resolution* rule, applied to goal G4 and assertion A5, with $\{z \leftarrow a\}$, we obtain

	G6. $\boxed{not\ (a \prec a)}^{+}$

By the *resolution* rule, applied to the *irreflexivity* axiom

$\boxed{not\ (x \prec x)}^{-}$	

and goal G6, with $\{x \leftarrow a\}$, we obtain the final goal

	G7. *true*

Note that henceforth we do not indicate which of the dual forms (AA, AG, and so on) of the *resolution* or *equivalence* rule is being applied; this should be evident.

Because we have proved the validity of the *asymmetry* property, we may add it as an assertion in future proofs within the theory of strict partial orderings.

Example (family theory). In Section 6.1, we defined a theory of family relationships. In the "family" interpretation $\mathcal{I}$ we have in mind, the domain is the set of people, and, intuitively, for the function symbols f and m,

$f(x)$ is the father of x

$m(x)$ is the mother of x,

and, for the predicate symbols p, gf, and gm,

$p(x, y)$ means y is a parent of x

$gf(x, y)$ means y is a grandfather of x

$gm(x, y)$ means y is a grandmother of x.

We define the theory by the following set of axioms:

$(\forall x)p(x, f(x))$	(*father*)
$(\forall x)p(x, m(x))$	(*mother*)
$(\forall x, y)\big[\mathit{if}\ \ p(x, y)\ \ \mathit{then}\ \ gf\big(x, f(y)\big)\big]$	(*grandfather*)
$(\forall x, y)\big[\mathit{if}\ \ p(x, y)\ \ \mathit{then}\ \ gm\big(x, m(y)\big)\big]$	(*grandmother*)

That is, everyone's father or mother is his or her parent, and the father [mother] of one's parent is his or her grandfather [grandmother].

In this *family theory* we gave an informal argument to show the validity of the sentence

$$(\forall x)(\exists z)gm(x, z),$$

that is, everyone has a grandmother. We can now prove it as a theorem in the theory using the deductive-tableau system.

We begin with the tableau

assertions	goals
$p\big(x, f(x)\big)$ (*father*)	
$p\big(x, m(x)\big)$ (*mother*)	
$\mathit{if}\ \ p(x, y)$ $\mathit{then}\ \ gf\big(x, f(y)\big)$ (*grandfather*)	
$\mathit{if}\ \ p(x, y)$ $\mathit{then}\ \ gm\big(x, m(y)\big)$ (*grandmother*)	
	G1. $(\forall x)^{\forall}(\exists z)^{\exists}gm(x, z)$

By the $\forall$-*elimination* and $\exists$-*elimination* rules, we may drop the quantifiers of goal G1, to obtain

	G2. $\boxed{gm(a, z)}^{+}$

The bound variable x, whose quantifier is of universal force in goal G1, is replaced by the skolem constant a in forming goal G2.

Applying the *resolution* rule to the *grandmother* axiom

if $p(x, y)$ *then* $\boxed{gm(x, m(y))}^-$	

and goal G2, with $\{x \leftarrow a,\ z \leftarrow m(y)\}$, we derive the goal

	G3. $\boxed{p(a, y)}^+$

By the *resolution* rule, applied to the *father* axiom

$\boxed{p(x, f(x))}^-$	

and goal G3, with $\{x \leftarrow a,\ y \leftarrow f(a)\}$, we obtain the final goal

	G4. *true*

$\blacksquare$

The reader may observe that the deductive-tableau proof reflects the informal reasoning given in Section 6.1. In **Problem 7.1**, the reader is requested to carry out another deductive-tableau proof in the family theory. In **Problem 7.2**, a proof in an augmented family theory is requested. In **Problem 7.3**, a deductive-tableau proof in a different axiomatic theory is requested.

If a theory is defined by an infinite set of axioms, we do not include all the axioms as assertions, because each tableau can have only a finite number of assertions. In the case of a theory with equality, we extend the system instead by introducing a new deduction rule that takes the place of infinitely many axioms.

7.2 EQUALITY RULE

If we want to prove the validity of a sentence S in a theory with equality, the most straightforward approach would be to add the equality axioms as assertions of our initial tableau. We have three simple axioms and two axiom schemata. The three axioms are

$(\forall\, x,\, y,\, z)\begin{bmatrix} \textit{if} \;\; x = y \;\; \textit{and} \;\; y = z \\ \textit{then} \;\; x = z \end{bmatrix}$	(*transitivity*)
$(\forall\, x,\, y)\begin{bmatrix} \textit{if} \;\; x = y \\ \textit{then} \;\; y = x \end{bmatrix}$	(*symmetry*)
$(\forall\, x)[x = x]$	(*reflexivity*)

The two axiom schemata are

For every k-ary function symbol f and for each i from 1 through k,

$$\begin{matrix} (\forall\, x,\, y) \\ (\forall\, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k) \end{matrix} \begin{bmatrix} \textit{if} \;\; x = y \\ \textit{then} \;\; f(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_k) = \\ \qquad f(z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_k) \end{bmatrix}$$

(*functional substitutivity for* f)

For every ℓ-ary predicate symbol q (other than $=$) and for each j from 1 through ℓ,

$$\begin{matrix} (\forall\, x,\, y) \\ (\forall\, z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_\ell) \end{matrix} \begin{bmatrix} \textit{if} \;\; x = y \\ \textit{then} \;\; q(z_1, \ldots, z_{j-1}, x, z_{j+1}, \ldots, z_\ell) \\ \equiv \\ q(z_1, \ldots, z_{j-1}, y, z_{j+1} \ldots, z_\ell) \end{bmatrix}$$

(*predicate substitutivity for* q)

The *functional-substitutivity* axiom schema actually represents an infinite set of axioms: one for every k-ary function symbol f and for each i from 1 through k. Similarly, the *predicate-substitutivity* axiom schema represents an infinite set of axioms: one for every ℓ-ary predicate symbol q (other than $=$) and for each j from 1 through ℓ.

This approach fails because we can only have finitely many assertions in a tableau. A more practical approach is to drop the *symmetry*, *transitivity*, *functional-substitutivity*, and *predicate-substitutivity* assertions altogether (leaving only the *reflexivity* assertion) and introduce instead an *equality* rule, which resembles the *equivalence* rule for predicate logic and enables us to treat equality in an efficient way.

THE BASIC FORM

The basic form of the equality rule is expressed as follows.

Rule (AA-equality, left-to-right)

assertions	goals
$\mathcal{A}_1[\overline{s=t}]$	
$\mathcal{A}_2\langle\overline{s'}\rangle$	
$\mathcal{A}_1\theta[\overline{\mathit{false}}]$ *or* $\mathcal{A}_2\theta\langle\overline{t\theta}\rangle$	

where

- $\overline{s=t}$ stands for the free, quantifier-free equalities $s_1 = t_1, \ldots, s_k = t_k$ $(k \geq 1)$, which occur in $\mathcal{A}_1$.
- $\overline{s'}$ stands for the free, quantifier-free subterms $s'_1, \ldots, s'_\ell$ $(\ell \geq 1)$, which occur in $\mathcal{A}_2$.
- The free variables of $\mathcal{A}_1[\overline{s=t}]$ and $\mathcal{A}_2\langle\overline{s'}\rangle$ are renamed so that the rows have no free variables in common.
- $\overline{t\theta}$ stands for $t_1\theta, \ldots, t_k\theta$.
- θ is a most-general separate-unifier for the tuple of subterms $\langle\overline{s}, \overline{s'}\rangle$ and the tuple of subterms $\langle\overline{t}\rangle$; that is, $s_1\theta, \ldots, s_k\theta$ and $s'_1\theta, \ldots, s'_\ell\theta$ are all identical terms and $t_1\theta, \ldots, t_k\theta$ are all identical terms. The terms $s_1\theta, \ldots, s_k\theta, s'_1\theta, \ldots, s'_\ell\theta$ must be distinct from the terms $t_1\theta, \ldots, t_k\theta$.
- $\overline{\mathit{false}}$ stands for *false*, ..., *false* (k times). ┛

More precisely, to apply the *AA-equality* rule to two assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ of a tableau:

- Rename the free variables of $\mathcal{A}_1$ and $\mathcal{A}_2$ if necessary to ensure that they have no free variables in common.
- Select free, quantifier-free subsentences of $\mathcal{A}_1$,

  $$\overline{s=t}: \quad s_1 = t_1, \quad \ldots, \quad s_k = t_k \qquad (k \geq 1),$$

  and free, quantifier-free subterms of $\mathcal{A}_2$,

  $$\overline{s'}: \quad s'_1, \quad \ldots, \quad s'_\ell \qquad (\ell \geq 1),$$

  such that θ is a most-general separate-unifier of the tuples $\langle s_1, \ldots, s_k, s'_1, \ldots, s'_\ell\rangle$ and $\langle t_1, \ldots, t_k\rangle$; that is,

  - $s_1\theta, \ldots, s_k\theta$ and $s'_1\theta, \ldots, s'_\ell\theta$ are all the same term, which we shall call $s\theta$.

- $t_1\theta, \ \ldots, \ t_k\theta$ are all the same term, which we shall call $t\theta$. We require that $s\theta$ and $t\theta$ be distinct. (Note that we invoke the *separate-tuple-unification* algorithm of Section 4.8.)

- Apply θ safely to the assertion $\mathcal{A}_1$ and replace all free occurrences of $s\theta = t\theta$ in $\mathcal{A}_1\theta$ with *false*, obtaining the disjunct
$$\mathcal{A}_1\theta[\overline{\mathit{false}}].$$
- Apply θ safely to the assertion $\mathcal{A}_2$ and replace safely one or more occurrences of $s\theta$ in $\mathcal{A}_2\theta$ with $t\theta$, obtaining the disjunct
$$\mathcal{A}_2\theta\langle\overline{t\theta}\rangle.$$
- Simplify the disjunction
$$\mathcal{A}_1\theta[\overline{\mathit{false}}] \ \ \mathit{or} \ \ \mathcal{A}_2\theta\langle\overline{t\theta}\rangle.$$
- Add the simplified disjunction to the tableau as a new assertion.

Remark (at least one replacement). Although the substitution notation admits the possibility that no equality $\overline{s = t}$ actually occurs in $\mathcal{A}_1$, the wording of the rule requires that some equalities actually do occur. Similarly, we require that at least one subterm of $\mathcal{A}_2\theta$ actually be replaced, even though the notation does not imply this. Otherwise, there would be no point in applying the rule. For the same reason, we do not apply the rule if $s\theta$ and $t\theta$ are identical. ▟

The *equality* rule allows us to drop the *symmetry*, *transitivity*, *functional-substitutivity*, and *predicate-substitutivity* axioms from our tableau; we must still retain the simple *reflexivity* axiom as an initial assertion

$x = x$	

We also have the following *right-to-left version* of the *equality* rule.

Rule (AA-equality, right-to-left)

assertions	goals
$\mathcal{A}_1[\overline{s = t}]$	
$\mathcal{A}_2\langle\overline{t'}\rangle$	
$\mathcal{A}_1\theta[\overline{\mathit{false}}]$ or $\mathcal{A}_2\theta\langle\overline{s\theta}\rangle$	

where $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the restrictions analogous to those in the left-to-right version of the rule. ▟

This rule allows us to replace occurrences of $t\theta$ in $\mathcal{A}_2\theta$ with $s\theta$, rather than the other way around.

POLARITY

The *polarity* strategy for the *AA-equivalence* rule should also be applied to both versions of the *equality* rule.

Strategy (polarity)

An application of the *AA-equality* rule is in accordance with the *polarity strategy* if

at least one of the occurrences of

$$s_1 = t_1, \quad \ldots, \quad s_k = t_k$$

in $\mathcal{A}_1$, whose instances are replaced by *false* in applying the rule, is of negative polarity in the tableau. ┘

In the *polarity* strategy, the negative polarity need not be strict; the occurrence in question may actually have both polarities. Note that the *polarity* strategy places no restriction on the subterms $s'_1, \ldots, s'_\ell$ of $\mathcal{A}_2$.

Example. Suppose our tableau contains the assertions

assertions	goals
$\mathcal{A}_1$: *if* $r(x)$ *then* $\left[\boxed{f(x,\,a)} = g(x,\,y)\right]^-$	
$\mathcal{A}_2$: $q\left(\boxed{f(b,\,z)},\,y,\,z\right)$	

Let us apply the *AA-equality* rule, left-to-right, to $\mathcal{A}_1$ and $\mathcal{A}_2$. The subterms to be matched are indicated by boxes.

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the free variable y in common. We therefore rename y as $\widehat{y}$ in $\mathcal{A}_2$, to obtain the assertion

$$\widehat{\mathcal{A}_2}: \quad q\left(\boxed{f(b,\,z)},\,\widehat{y},\,z\right).$$

Consider the free subsentence

$$s = t: \quad f(x,\,a) = g(x,\,y)$$

in $\mathcal{A}_1$ and the free subterm

$$s': \quad f(b,\,z)$$

in $\widehat{\mathcal{A}_2}$. The terms

$$s: \quad f(x, a) \qquad \text{and} \qquad s': \quad f(b, z)$$

are unifiable under the most-general unifier

$$\theta: \ \{x \leftarrow b, \ z \leftarrow a\};$$

the unified terms $s\theta$ and $s'\theta$ are identical to $f(b, a)$.

Applying θ to $\mathcal{A}_1$, we obtain

$$\mathcal{A}_1\theta: \quad \textit{if} \ \ r(b) \ \ \textit{then} \ \ f(b, a) = g(b, y),$$

where the unified equality is

$$(s = t)\theta: \quad f(b, a) = g(b, y).$$

Replacing the equality $f(b, a) = g(b, y)$ with the truth symbol *false* in $\mathcal{A}_1\theta$, we obtain

$$\mathcal{A}_1^*: \quad \textit{if} \ \ r(b) \ \ \textit{then} \ \ \textit{false}.$$

Applying θ to $\widehat{\mathcal{A}_2}$, we obtain

$$\widehat{\mathcal{A}_2}\theta: \quad q\big(f(b, a), \ \widehat{y}, \ a\big).$$

Replacing the subterm $s\theta : f(b, a)$ with $t\theta : g(b, y)$ in $\widehat{\mathcal{A}_2}\theta$, we obtain

$$\mathcal{A}_2^*: \quad q\big(g(b, y), \ \widehat{y}, \ a\big).$$

Forming the disjunction $(\mathcal{A}_1^* \ \textit{or} \ \mathcal{A}_2^*)$, we obtain

$$\begin{array}{l} \textit{if} \ \ r(b) \ \ \textit{then} \ \ \textit{false} \\ \quad \textit{or} \\ q\big(g(b, y), \ \widehat{y}, \ a\big). \end{array}$$

This reduces (under *true-false* simplification) to the new assertion

$\textit{not} \ r(b)$ $\quad \textit{or}$ $q\big(g(b, y), \ \widehat{y}, \ a\big)$	

which is added to the tableau.

As usual, we do not add the intermediate sentences $\widehat{\mathcal{A}_2}$, $\mathcal{A}_1\theta$, $\widehat{\mathcal{A}_2}\theta$, $\mathcal{A}_1^*$, $\mathcal{A}_2^*$, or the unsimplified disjunction to the tableau.

This application of the rule is in accordance with the *polarity* strategy because the occurrence of the equality

$$f(x, a) = g(x, y)$$

in $\mathcal{A}_1$ is negative in the tableau. ■

JUSTIFICATION

To justify the *AA-equality* rule, we show that the old (given) tableau is equivalent to the new tableau. The justification of the *AA-equality* rule is analogous to that of the *AA-equivalence* rule. We justify the left-to-right version of the rule. Because the rule preserves equivalence, we can use the *special justification* proposition (Section 5.2), rather than the *general justification* proposition, in the proof.

Justification (AA-equality). Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two assertions that satisfy the restrictions for applying the *AA-equality* rule. As in the justifications for the *AA-resolution* and *AA-equivalence* rules, we may assume that the free variables of $\mathcal{A}_1$ have been renamed as necessary to ensure that the assertions have no free variables in common.

Let $\mathcal{I}$ be an interpretation under which the required tableau $\mathcal{T}_r$ is false; that is, the universal closures of the required assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are both true under $\mathcal{I}$. By the *special justification* proposition, it suffices to establish that the generated tableau $\mathcal{T}_g$ is false under $\mathcal{I}$, that is, that the universal closure of the generated assertion, the disjunction

$$\begin{array}{c} \mathcal{A}_1\theta[\overline{false}] \\ or \\ \mathcal{A}_2\theta\langle\overline{t\theta}\rangle, \end{array}$$

is also true under $\mathcal{I}$.

Because the universal closures of $\mathcal{A}_1$ and $\mathcal{A}_2$ are true under $\mathcal{I}$, the universal closures of $\mathcal{A}_1\theta$ and $\mathcal{A}_2\theta$ are also true under $\mathcal{I}$, by the *universal closure-instantiation* proposition. Therefore, by the *semantic-rule-for-universal-closure* proposition, $\mathcal{A}_1\theta$ and $\mathcal{A}_2\theta$ are themselves true under any interpretation that agrees with $\mathcal{I}$ on the constant, function, and predicate symbols of $\mathcal{A}_1\theta$ and $\mathcal{A}_2\theta$. Let $\mathcal{I}'$ be any such interpretation.

The proof now distinguishes between two cases.

Case: $(s = t)\theta$ is false under $\mathcal{I}'$

That is, the equivalence $(s = t)\theta \equiv false$ is true under $\mathcal{I}'$. Then (by the substitutivity of equivalence), because $\mathcal{A}_1\theta$ is true under $\mathcal{I}'$,

$$\mathcal{A}_1\theta[\overline{false}]$$

is also true under $\mathcal{I}'$. It follows (by the semantic rule for the *or* connective) that the disjunction

$$\begin{array}{c} \mathcal{A}_1\theta[\overline{false}] \\ or \\ \mathcal{A}_2\theta\langle\overline{t\theta}\rangle \end{array}$$

is also true under $\mathcal{I}'$.

Case: $(s = t)\theta$ is true under $\mathcal{I}'$

That is, $s\theta = t\theta$ is true under $\mathcal{I}'$. Hence, by the *substitutivity-of-equality* proposition (Section 6.5), because $\mathcal{A}_2\theta$ is true under $\mathcal{I}'$,

$$\mathcal{A}_2\theta\langle\, \overline{t\theta}\,\rangle$$

is also true under $\mathcal{I}'$. It follows (by the semantic rule for the *or* connective) that the disjunction

$$\begin{array}{l} \mathcal{A}_1\theta\,[\,\overline{\mathit{false}}\,] \\ \quad \mathit{or} \\ \mathcal{A}_2\theta\langle\, \overline{t\theta}\,\rangle \end{array}$$

is true under $\mathcal{I}'$.

In each case, we have concluded that the generated assertion, the disjunction

$$\begin{array}{l} \mathcal{A}_1\theta\,[\,\overline{\mathit{false}}\,] \\ \quad \mathit{or} \\ \mathcal{A}_2\theta\langle\, \overline{t\theta}\,\rangle, \end{array}$$

is true under $\mathcal{I}'$, for any interpretation $\mathcal{I}'$ that agrees with $\mathcal{I}$ on the constant, function, and predicate symbols of $\mathcal{A}_1\theta$ and $\mathcal{A}_2\theta$, and hence of the new assertion. Therefore (by the *semantic-rule-for-universal-closure* proposition), the universal closure of the generated assertion is true under $\mathcal{I}$, as we wanted to prove. The final simplification step preserves equivalence. $\lrcorner$

The justification of the right-to-left version of the rule can be shown by the symmetry of equality.

DUAL FORMS

We have given the AA-form of the rule, which applies to two assertions. By duality, we can introduce forms of the rule that apply to an assertion and a goal, or to two goals. We present only the AG-form (left-to-right version), which applies to an assertion and a goal. This is the most commonly used form.

Rule (AG-equality, left-to-right)

assertions	goals
$\mathcal{A}[\,\overline{s = t}\,]$	
	$\mathcal{G}\langle\, \overline{s'}\,\rangle$
	$\mathit{not}\ (\mathcal{A}\theta[\,\overline{\mathit{false}}\,])$ and $\mathcal{G}\theta\langle \overline{t\theta}\rangle$

where $\mathcal{A}$ and $\mathcal{G}$ satisfy the same conditions as $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, in the *AA-equality* rule. ┘

There are analogous right-to-left forms. The justification of the dual forms of the *equality* rule follows by duality.

The *polarity* strategy for each dual form of the rule is analogous to that for the *AA-equality* rule: at least one of the free occurrences of $s_1 = t_1, \ldots, s_k = t_k$, which are replaced by *false* in applying the rule, should be negative in the tableau.

Example. Suppose our tableau contains the assertion and goal

assertions	goals
$\mathcal{A}: \left[f(x, y) = \boxed{g(x, a)}\right]^-$	
	$\mathcal{G}: \begin{array}{c}(\forall y)\, q\left(\boxed{g(b, z)}, y, z\right) \\ \textit{and} \\ p\left(\boxed{g(u, a)}\right)\end{array}$

Let us apply the *AG-equality* rule, right-to-left, to assertion $\mathcal{A}$ and goal $\mathcal{G}$. The subterms to be matched are indicated with boxes. Note that $\mathcal{A}$ and $\mathcal{G}$ have no free variables in common.

Consider the free subsentence

$$s = t: \quad f(x, y) = g(x, a)$$

in $\mathcal{A}$, which has negative polarity in the tableau, and consider the free subterms

$$t': \ g(b, z) \qquad \text{and} \qquad t'': \ g(u, a)$$

in $\mathcal{G}$.

The terms

$$t: \ g(x, a), \quad t': \ g(b, z), \quad \text{and} \quad t'': \ g(u, a)$$

are unifiable, with a most-general unifier

$$\theta: \ \{x \leftarrow b,\ z \leftarrow a,\ u \leftarrow b\};$$

the unified term is $g(b, a)$.

We apply θ to the assertion $\mathcal{A}$ and the goal $\mathcal{G}$, obtaining

$$\mathcal{A}\theta: \ f(b, y) = g(b, a)$$

and

$$\mathcal{G}\theta: \quad \begin{array}{l} (\forall y)q\big(g(b, a),\, y,\, a\big) \\ \quad \textit{and} \\ p\big(g(b, a)\big), \end{array}$$

respectively. Replacing the equality

$$(s = t)\theta: \quad f(b, y) = g(b, a)$$

in $\mathcal{A}\theta$ with *false*, and (safely) replacing the two subterms

$$t\theta: \quad g(b, a)$$

in $\mathcal{G}\theta$ with

$$s\theta: \quad f(b, y),$$

we obtain the new goal

$$\begin{array}{l} \textit{not false} \\ \quad \textit{and} \\ (\forall y')\, q\big(f(b, y),\, y',\, a\big). \\ \quad \textit{and} \\ p\big(f(b, y)\big). \end{array}$$

Note that we have used the rule to replace two occurrences of $t\theta$ with $s\theta$. Also, we have renamed the variable y of the quantifier $(\forall y)$ as y' to avoid capturing the free occurrence of y in $f(b, y)$.

The new goal reduces (by *true-false* simplification) to

	$(\forall y')\, q\,\big(f(b, y),\, y',\, a\big)$ *and* $p\big(f(b, y)\big)$

Remark (skolemization in axiomatic theories). The skolemization process has been described and justified for pure predicate logic. We have been applying the process, however, to remove quantifiers in axiomatic theories. Removing a quantifier of both forces or a quantifier of strict existential force presents no problem because these phases of the process preserve equivalence; if two sentences are equivalent in predicate logic, they are equivalent in any axiomatic theory.

The trouble arises when we attempt to remove quantifiers of strict universal force, because this phase does not preserve equivalence, but only validity; a process that preserves validity in predicate logic may not preserve validity in an axiomatic theory. In predicate logic, to be valid a sentence must be true under all interpretations; in a theory, to be valid a sentence must be true only under the models of the theory. Therefore the removal of quantifiers of universal force requires special justification for each theory we consider. We shall not do this here, but only indicate why it must be done.

For a finite theory, the justification of this phase of the skolemization process is straightforward. We choose as our skolem function (or skolem constant) a symbol that does not occur in any of the axioms.

This approach does not suffice if our theory is infinite and defined by one or more axiom schemata. In this case, it may happen that every constant and function symbol occurs in some instance of an axiom schema.

For example, in the theory of equality, suppose we remove a quantifier of strict universal force by introducing a new unary skolem function symbol f. In that case, we automatically provide the appropriate instance of the *functional substitutivity* axiom schema,

$$(\forall x, y)\begin{bmatrix} \textit{if} \;\; x = y \\ \textit{then} \;\; f(x) = f(y) \end{bmatrix} \qquad \textit{(functional substitutivity for } f\textit{)}$$

In an infinite theory, the danger we face is that, in removing a quantifier of strict universal force from a sentence that is not valid, we introduce a skolem function symbol f and obtain a sentence that is valid in the theory, because it is a consequence of some instances of our axiom schemata that refer to f. This turns out to be impossible in the theory of equality or any of the other theories we shall consider. Therefore we shall use the skolemization process freely in these theories. ⌟

In **Problem 7.4**, the reader is requested to show that the *transitivity*, *symmetry*, and *substitutivity* properties of equality can actually be proved in a tableau with the *equality* rule. In **Problem 7.5**, the reader is asked to prove the validity of one sentence in the theory of equality and to show the nonvalidity of another sentence.

7.3 FINITE THEORIES WITH EQUALITY

A tableau that includes among its initial assertions the *reflexivity* axiom and to which we may apply the *equality* rule, as well as any of the predicate-logic rules, will be called a *tableau with equality*.

We have seen that we can prove a sentence S within a particular finite theory by adding the axioms for the theory as the initial assertions in a predicate-logic tableau, and adding S as the initial goal. In the same way, if we want to prove a sentence S in a finite theory with equality (that is, one defined by the equality axioms plus a finite set of special axioms), we may add the special axioms as the initial assertions of a tableau with equality that has S as its initial goal.

In the following sections, we apply the deductive-tableau framework to prove the validity of sentences in two finite theories with equality. We start with the theory of weak partial orderings.

THEORY OF WEAK PARTIAL ORDERINGS

We have defined earlier (Section 6.6) the theory of a weak partial ordering $\preceq$ as the theory with equality whose special axioms are

$$(\forall x, y, z)\begin{bmatrix}\textit{if } x \preceq y \textit{ and } y \preceq z\\ \textit{then } x \preceq z\end{bmatrix} \qquad (\textit{transitivity})$$

$$(\forall x, y)\begin{bmatrix}\textit{if } x \preceq y \textit{ and } y \preceq x\\ \textit{then } x = y\end{bmatrix} \qquad (\textit{antisymmetry})$$

$$(\forall x)[x \preceq x] \qquad (\textit{reflexivity})$$

To prove the validity of a sentence $\mathcal{S}$ in the theory of the weak partial ordering $\preceq$ within the tableau framework, we need only prove the following tableau with equality:

assertions	goals
if $x \preceq y$ *and* $y \preceq z$ *then* $x \preceq z$ (*transitivity*)	
if $x \preceq y$ *and* $y \preceq x$ *then* $x = y$ (*antisymmetry*)	
$x \preceq x$ (*reflexivity*)	
	$\mathcal{S}$

Here again, by outermost skolemization, we have dropped the outermost universal quantifiers from the initial assertions.

Recall that, because this is a tableau with equality, we also include the *reflexivity* axiom ($x = x$) among our initial assertions, and during the proof we may apply the *equality* rule (both the left-to-right and right-to-left versions) as well as the other predicate-logic rules. We need not include the axioms for equality (other than *reflexivity*) as assertions in the tableau.

Example (irreflexive restriction). Consider the augmented theory formed by adding to the theory of the weak partial ordering $\preceq$ the following axiom, which defines the *irreflexive restriction* $\prec$ associated with $\preceq$:

$$(\forall x, y)\begin{bmatrix} x \prec y \\ \equiv \\ x \preceq y \ \textit{and} \ \textit{not}\ (x = y)\end{bmatrix} \qquad \textit{(irreflexive restriction)}$$

It can then be shown that $\prec$ is indeed a strict partial ordering, i.e., that $\prec$ is transitive and irreflexive. This is stated in the *irreflexive-restriction* proposition of the theory of the weak partial ordering $\preceq$ (Section 6.6). We show the irreflexivity of $\prec$ here; its transitivity is left as an exercise (**Problem 7.6**).

Suppose we would like to show the validity of the *irreflexivity* property

$$(\forall x)\left[\textit{not}\ (x \prec x)\right] \qquad \textit{(irreflexivity)}$$

in this theory.

We begin with a tableau over the theory of weak partial orderings that contains, in addition to the *reflexivity* axiom for equality and the axioms for a weak partial ordering, the definition of the irreflexive-restriction relation

assertions	goals
$\begin{bmatrix} \boxed{x \prec y} \\ \equiv \\ x \preceq y \ \textit{and} \ \textit{not}\ (x = y)\end{bmatrix}^{-}$ *(irreflexive restriction)*	

as an initial assertion and the desired *irreflexivity* property,

	G1. $(\forall x)^{\forall}[\textit{not}\ (x \prec x)]$

as its initial goal.

By the $\forall$-*elimination* rule, we may drop the quantifier $(\forall x)^{\forall}$ from the initial goal G1, replacing the bound variable x with the skolem constant a, to obtain

	G2. $\textit{not}\ \boxed{a \prec a}$

Applying the *equivalence* rule to the *irreflexive restriction* axiom and goal G2, with $\{x \leftarrow a,\ y \leftarrow a\}$, we obtain the goal

	G3. $\textit{not}\ \left(a \preceq a \ \textit{and} \ \textit{not}\ \boxed{a = a}^{+}\right)$

Applying the *resolution* rule to the *reflexivity* axiom for equality,

$\boxed{x = x}^{-}$	

and goal G3, we obtain the final goal

	G4. *true*

In **Problem 7.7** the reader is requested to provide a tableau proof of the *reflexive-closure* proposition, i.e., that in a theory with equality defined by the axioms for a strict partial ordering, the reflexive closure $\preceq$ of $\prec$ is a weak partial ordering (Section 6.8).

Remark (reflexivity of equality). In the final step of the previous example, we applied the *resolution* rule to goal G3 and the *reflexivity* axiom $(x = x)$, to obtain the goal G4. Resolution with the *reflexivity* axiom is a frequent step in proofs in theories with equality. It will be convenient for us to apply this step automatically whenever a positive subsentence of the form $(t = t)$ appears in an assertion or a goal. For brevity we may then say that goal G4 is obtained from goal G3 "by the reflexivity of equality," without giving the assertion or mentioning the resolution step. We shall still annotate the subsentence $(a = a)$ in G3 with a box indicating its positive polarity.

If a positive subsentence of form $(t = t')$ appears, where t and t' are unifiable but not identical, we shall not apply the resolution step automatically. Also, we shall mention the rule, the assertion, and the most-general unifier explicitly in this case.

THEORY OF GROUPS

We have defined the theory of groups (Section 6.7) as the theory with equality whose special axioms are

$(\forall x)\big[x \circ e \;=\; x\big]$	*(right identity)*
$(\forall x)\big[x \circ x^{-1} \;=\; e\big]$	*(right inverse)*
$(\forall x, y, z)\Big[(x \circ y) \circ z \;=\; x \circ (y \circ z)\Big]$	*(associativity)*

To prove the validity of a sentence $\mathcal{S}$ in this theory, we must prove the following tableau with equality:

assertions		goals
$x \circ e = x$	(*right identity*)	
$x \circ x^{-1} = e$	(*right inverse*)	
$(x \circ y) \circ z = x \circ (y \circ z)$	(*associativity*)	
		$\mathcal{S}$

Again, because this is a tableau with equality, it includes implicitly the *reflexivity* axiom ($x = x$) among its assertions, and during the proof we may apply the *equality* rule, as well as the other predicate-logic deduction rules.

Example (right cancellation). Suppose we would like to prove the validity of the property

$$(\forall x, y, z) \begin{bmatrix} \textit{if} \;\; x \circ z = y \circ z \\ \textit{then} \;\; x = y \end{bmatrix} \qquad (\textit{right cancellation})$$

in the theory of groups. Our deductive-tableau proof resembles the informal proof of the same proposition in Section 6.7.

We consider the initial goal

assertions	goals
	G1. $(\forall x, y, z)^{\forall} \begin{bmatrix} \textit{if} \;\; x \circ z = y \circ z \\ \textit{then} \;\; x = y \end{bmatrix}$

By the $\forall$-*elimination* rule, we may drop the quantifiers from goal G1, replacing the bound variables x, y, and z with skolem constants a, b, and c, respectively, to obtain

	G2. $\textit{if} \;\; a \circ c = b \circ c$ $\textit{then} \;\; a = b$

By the *if-split* rule, we decompose goal G2 into the assertion and goal

A3. $\left[\boxed{a \circ c} = b \circ c\right]^{-}$	
	G4. $\boxed{a} = b$

Applying the *equality* rule (right-to-left) to the *right-identity* axiom,

$\left[x \circ e \;=\; \boxed{x}\right]^{-}$	

and goal G4, with $\{x \leftarrow a\}$, we replace a with $a \circ e$ in the goal, to obtain

	G5. $a \circ e \;=\; \boxed{b}$

Applying the *equality* rule (right-to-left) once more to the *right-identity* axiom and goal G5, with $\{x \leftarrow b\}$, we replace b with $b \circ e$ in the goal, to obtain

	G6. $a \circ \boxed{e} \;=\; b \circ e$

Applying the *equality* rule (right-to-left) to the *right-inverse* axiom,

$\left[x \circ x^{-1} = \boxed{e}\right]^{-}$	

with $\{\,\}$, we replace the annotated occurrence of e in goal G6 with $x \circ x^{-1}$, to obtain

	G7. $a \circ (x \circ x^{-1}) \;=\; b \circ \boxed{e}$

We would like to apply the *equality* rule again to the *right-inverse* axiom and goal G7; these rows, however, have the free variable x in common. We rename the variables in these rows to avoid this coincidence, to obtain

$\left[x_1 \circ x_1^{-1} = \boxed{e}\right]^{-}$	
	G7′. $a \circ (x_2 \circ x_2^{-1}) \;=\; b \circ \boxed{e}$

(To avoid future renaming, we have actually renamed x in both rows.)

Now we may apply the *equality* rule (right-to-left) to replace e with $x_1 \circ x_1^{-1}$ in goal G7′, obtaining

	G8. $\boxed{a \circ (x_2 \circ x_2^{-1})} = \boxed{b \circ (x_1 \circ x_1^{-1})}$

By two applications of the *equality* rule (right-to-left) to the *associativity* axiom,

$\left[(x \circ y) \circ z = \boxed{x \circ (y \circ z)}\right]^-$	

and goal G8, with $\{x \leftarrow a,\ y \leftarrow x_2,\ z \leftarrow x_2^{-1}\}$ and then with $\{x \leftarrow b,\ y \leftarrow x_1,\ z \leftarrow x_1^{-1}\}$, we may rewrite the goal as

	G9. $\left(\boxed{a \circ x_2}\right) \circ x_2^{-1} = (b \circ x_1) \circ x_1^{-1}$

By the *equality* rule, applied to assertion A3 and goal G9, with $\{x_2 \leftarrow c\}$, we replace $a \circ x_2$ with $b \circ c$ in the goal, to obtain

	G10. $\boxed{(b \circ c) \circ c^{-1} = (b \circ x_1) \circ x_1^{-1}}^+$

At last, applying the *resolution* rule to the *reflexivity* axiom $(x = x)$ and goal G10, with $\{x_1 \leftarrow c,\ x \leftarrow (b \circ c) \circ c^{-1}\}$, we obtain the final goal

	G11. *true*

Now that we have proved the *right-cancellation* property, we can add it as an assertion to the tableau of any subsequent proof in the theory of groups. ▪

In **Problem 7.8** the reader is requested to use the deductive-tableau technique to prove the following properties of the theory of groups:

$$(\forall x)\left[e \circ x = x\right] \qquad \textit{(left identity)}$$

$$(\forall x)\left[x^{-1} \circ x = e\right] \qquad \textit{(left inverse)}$$

$$(\forall x, y, z)\begin{bmatrix} \textit{if } z \circ x = z \circ y \\ \textit{then } x = y \end{bmatrix} \qquad \textit{(left cancellation)}$$

$$(\forall x)\begin{bmatrix} \textit{if } x \circ x = x \\ \textit{then } x = e \end{bmatrix} \qquad \textit{(nonidempotence)}$$

The following example illustrates how, within an axiomatic theory, we may define new functions by providing additional axioms.

Example (quotient). Suppose we define the quotient x/y of two elements x and y of a group by the following axiom:

$(\forall x, y)\big[x/y \ = \ x \circ y^{-1}\big]$	(*quotient*)

We would like to prove that the *cancellation* property holds, that is,

$$(\forall\, x,\, y)\big[(x/y) \circ y \ = \ x\big] \qquad (\textit{cancellation})$$

We attempt to prove the initial tableau over the theory of groups,

assertions	goals
$\Big[\boxed{x/y} \ = \ x \circ y^{-1}\Big]^{-}$ (*quotient*)	
	G1. $(\forall\, x,\, y)^{\forall}\big[(x/y) \circ y \ = \ x\big]$

The *quotient* axiom for the quotient function is included among the assertions; because the tableau is over the theory of groups, the group axioms and previously proved group theorems are also present.

By the $\forall$-*elimination* rule, we may drop the quantifiers of goal G1, to obtain

	G2. $\boxed{a/b} \circ b \ = \ a$

By the *equality* rule, applied to the *quotient* axiom and the goal G2, with $\{x \leftarrow a,\ y \leftarrow b\}$, we obtain

	G3. $\boxed{(a \circ b^{-1}) \circ b} \ = \ a$

By the *equality* rule, applied to the *associativity* axiom

$\Big[\boxed{(x \circ y) \circ z} \ = \ x \circ (y \circ z)\Big]^{-}$	

and goal G3, with $\{x \leftarrow a,\ y \leftarrow b^{-1},\ z \leftarrow b\}$, we obtain

	G4. $a \circ \left(\boxed{b^{-1} \circ b}\right) = a$

By the *equality* rule, applied to the *left-inverse* property

$\left[\boxed{x^{-1} \circ x} = e\right]^-$	

and goal G4, with $\{x \leftarrow b\}$, we obtain

	G5. $\boxed{a \circ e = a}^+$

Applying the *resolution* rule to the *right-identity* axiom

$\boxed{x \circ e = x}^-$	

and goal G5, with $\{x \leftarrow a\}$, we obtain the final goal

	G6. *true*

Note that the proof of the *cancellation* property depends on the proof of the *left-inverse* property, which was requested as an exercise. Had we attempted to prove the *cancellation* theorem without having proved the other theorem first, the proof, of course, would have been more cumbersome.

In **Problem 7.9** we interchange the roles of the *quotient* axiom and *cancellation* property. We assume that the quotient function x/y is defined alternatively by the axiom

$$(\forall\, x,\, y)\big[(x/y) \circ y \;=\; x\big] \qquad (\textit{cancellation})$$

and ask the reader to prove that the quotient x/y is then the same as $x \circ y^{-1}$, that is,

$$(\forall\, x,\, y)\big[x/y \;=\; x \circ y^{-1}\big] \qquad (\textit{quotient})$$

PROBLEMS

Use the deductive-tableau technique to carry out the following proofs.

Problem 7.1 (family theory) page 352

In the family theory, prove that

if Alice is the parent of her own father,
then Alice's father is his own grandfather.

An informal argument to show this was requested in Problem 6.1.

Problem 7.2 (augmented family theory) page 352

Suppose we augment the family theory with the following axiom:

$$(\forall\, x,\, y,\, z) \begin{bmatrix} \textit{if } p(x,\, z) \textit{ and } p(y,\, z) \\ \textit{then } s(x,\, y) \end{bmatrix} \qquad (\textit{sibling})$$

where $s(x,\, y)$ is intended to mean that x and y are siblings.

In this augmented theory, show that

If the mother of Bob is a parent of Alice
then Alice and Bob are siblings.

More precisely:

(a) Find a sentence $\mathcal{F}$ whose intuitive meaning is given by the English sentence above. Let the constants a and b denote Alice and Bob, respectively.

(b) Give a deductive-tableau proof of $\mathcal{F}$ in the family theory augmented by the *sibling* axiom.

Problem 7.3 (redhead) page 352

Suppose the *grandparent theory* is defined by the single axiom

$$(\forall\, x,\, z)\Big[gp(x,\, z) \;\equiv\; (\exists\, y)\big[p(x,\, y) \;\textit{ and }\; p(y,\, z)\big]\Big] \qquad (\textit{grandparent})$$

Intuitively, this means that z is a grandparent of x if and only if, for some person y, y is a parent of x and z is a parent of y. Within this theory, give a proof of the following sentence:

$$\begin{array}{l} \textit{if } \;(\exists\, x,\, z) \begin{bmatrix} gp(x,\, z) \;\textit{ and} \\ red(x) \;\textit{ and }\; \textit{not } red(z) \end{bmatrix} \\ \textit{then } \;(\exists\, x,\, z) \begin{bmatrix} p(x,\, z) \;\textit{ and} \\ red(x) \;\textit{ and }\; \textit{not } red(z) \end{bmatrix} \end{array} \qquad (\textit{redhead})$$

Intuitively, if $red(x)$ stands for "x is a redhead," this means that if some redhead has a nonredheaded grandparent, then some redhead has a nonredheaded parent.

Problem 7.4 (properties of equality) page 362

In a tableau with equality, show the following properties of equality:

(a) *Transitivity*

$$(\forall\, x,\, y,\, z)\begin{bmatrix} \textit{if}\ \ x = y\ \ \textit{and}\ \ y = z \\ \textit{then}\ \ x = z \end{bmatrix}$$

(b) *Symmetry*

$$(\forall\, x,\, y)\begin{bmatrix} \textit{if}\ \ x = y \\ \textit{then}\ \ y = x \end{bmatrix}$$

(c) *Functional Substitutivity*

$$(\forall\, x,\, y,\, z)\begin{bmatrix} \textit{if}\ x = y \\ \textit{then}\ f(x,\, z) = f(y,\, z) \end{bmatrix}$$

(d) *Predicate Substitutivity*

$$(\forall\, x,\, y,\, z)\begin{bmatrix} \textit{if}\ x = y \\ \textit{then}\ q(z,\, x) \equiv q(z,\, y) \end{bmatrix}.$$

Problem 7.5 (valid equality) page 362

Let $\mathcal{F}[x]$ stand for the sentence

$$\mathcal{F}[x]:\quad (\forall y,\, z)\begin{bmatrix} \textit{if}\ f\big(g(x)\big) = x \\ \textit{then}\ \textit{if}\ g(y) = g(z) \\ \qquad \textit{then}\ h\big(g(y),\, z\big) = h\big(g(z),\, y\big) \end{bmatrix}.$$

(a) In a deductive tableau with equality, prove the validity of $(\exists\, x)\mathcal{F}[x]$.

(b) Show that $(\forall\, x)\mathcal{F}[x]$ is not valid in the theory of equality.

Problem 7.6 (irreflexive restriction) page 364

In the theory of a weak partial ordering $\preceq$, prove that the irreflexive restriction $\prec$ associated with $\preceq$ is transitive, that is,

$$(\forall x,\, y,\, z)\begin{bmatrix} \textit{if}\ \ x \prec y\ \ \textit{and}\ \ y \prec z \\ \textit{then}\ \ x \prec z \end{bmatrix} \qquad \textit{(transitivity)}$$

Problem 7.7 (reflexive closure) page 365

Consider a theory with equality defined by the axioms of the theory of a strict partial ordering $\prec$. Prove that the corresponding *reflexive-closure* relation, defined by

$$(\forall x,\, y)\begin{bmatrix} x \preceq y \\ \equiv \\ x \prec y\ \ \textit{or}\ \ x = y \end{bmatrix} \qquad \textit{(reflexive closure)}$$

is a weak partial ordering, i.e., that the following properties hold:

(a) *Transitivity*

$$(\forall x, y, z)\begin{bmatrix} \textit{if} \;\; x \preceq y \;\; \textit{and} \;\; y \preceq z \\ \textit{then} \;\; x \preceq z \end{bmatrix}$$

(b) *Antisymmetry*

$$(\forall x, y)\begin{bmatrix} \textit{if} \;\; x \preceq y \;\; \textit{and} \;\; y \preceq x \\ \textit{then} \;\; x = y \end{bmatrix}$$

(c) *Reflexivity*

$$(\forall x)[x \preceq x].$$

Problem 7.8 (theory of groups) page 368

Prove the following properties of the theory of groups:

(a) *Left identity*

$$(\forall x)\big[e \circ x = x\big]$$

(b) *Left inverse*

$$(\forall x)\big[x^{-1} \circ x = e\big]$$

(c) *Left cancellation*

$$(\forall x, y, z)\begin{bmatrix} \textit{if} \;\; z \circ x = z \circ y \\ \textit{then} \;\; x = y \end{bmatrix}$$

(d) *Nonidempotence*

$$(\forall x)\begin{bmatrix} \textit{if} \;\; x \circ x = x \\ \textit{then} \;\; x = e \end{bmatrix}.$$

Problem 7.9 (quotient versus inverse) page 370

Suppose we define the quotient x/y of two elements x and y of a group by the following axiom:

$$(\forall x, y)\big[(x/y) \circ y = x\big] \qquad (\textit{cancellation})$$

Prove that the quotient x/y is then the same as $x \circ y^{-1}$, that is,

$$(\forall x, y)\big[x/y = x \circ y^{-1}\big] \qquad (\textit{quotient})$$